Control of Interface Evolution in Multi-Phase Fluid Flows

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Workshop on Numerical Methods for Optimal Control and Inverse Problems
Garching, 2013-03-13

Contents

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The Model

- $\rho_0 = \rho_1 \chi_{\Omega_1} + \rho_2 \chi_{\Omega_2}$ mixture of two immiscible viscous incompressible fluids in a bounded domain in $\mathbb{R}^2$.
- Multi-phase flow evolution by Navier–Stokes Eq. (cf. [Lions, 1996])

$$(NSE) \begin{cases} \rho \boldsymbol{y}_t + \rho[\boldsymbol{y} \cdot \nabla]\boldsymbol{y} - \mu \Delta \boldsymbol{y} + \nabla p = \rho \boldsymbol{u}, & \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \rho_t + [\boldsymbol{y} \cdot \nabla]\rho = 0, & \rho(0) = \rho_0, \\ \operatorname{div} \boldsymbol{y} = 0 & + B.C. \end{cases}$$

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Minimize

"Shape" "Geometry" "Cost"

$$J(\rho, \boldsymbol{u}) = \int_0^T \int_\Omega |\rho(t) - \sigma|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \frac{\beta}{2} \int_0^T \mathcal{H}^1(S_\rho) \, \mathrm{d}t + \frac{\alpha}{2} \int_0^T \int_\Omega |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$

subject to

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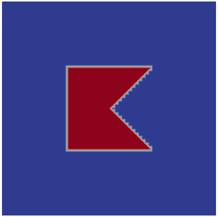
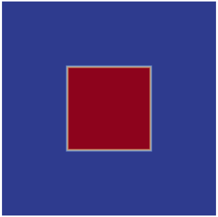
Evidence of the geometric functional

$\|\rho - \sigma\|^2_{L^2(\Omega_T)}$

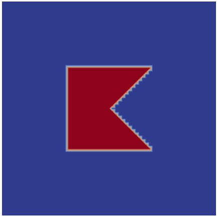


Target σ



$\|\rho - \sigma\|^2_{L^2(\Omega_T)}$

$+ \int_0^T \mathcal{H}^1(S_\rho)$



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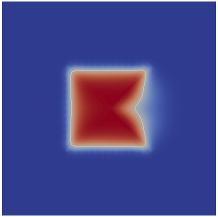

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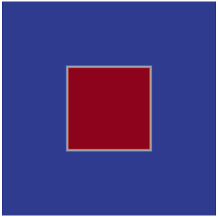

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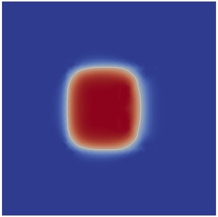

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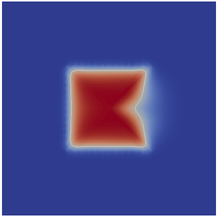
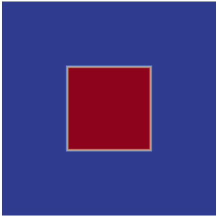


better corners

Target σ



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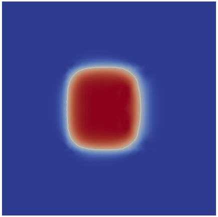
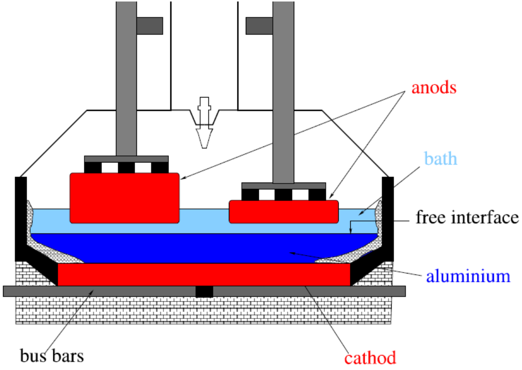


correct geometry

Application ([Gerbeau et al., 2006]): Aluminium production via electrolysis



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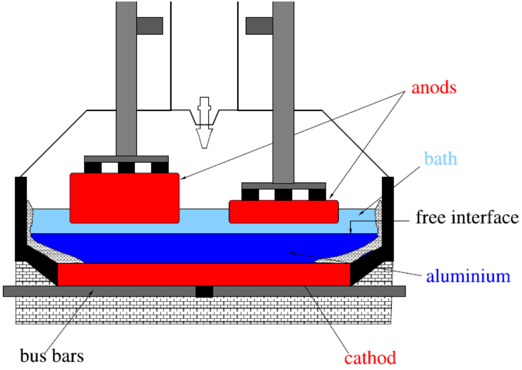

Anods shall not touch the interface!
$\Rightarrow$ Interface control

Goals

- Existence of optimum.
- (Necessary) first order optimality conditions.
- Numerical scheme with low order Finite Elements.
- Convergence of the numerical scheme.

Known result

- Optimization (analysis, no numerics) of L^2-functional (no geometric term) subject to Stokes equation, cf. [Kunisch and Lu, 2011].
- Convergent numerical scheme for equation (low regularity), cf. [Baňas and Prohl, 2010].

Contents

Analytical problems and strategy

Minimize

$$J(\rho, \boldsymbol{u}) = \int_0^T \int_\Omega |\rho(t) - \sigma|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \frac{\beta}{2} \int_0^T \mathcal{H}^1(S_\rho) \, \mathrm{d}t + \frac{\alpha}{2} \int_0^T \int_\Omega |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$

subject to

$$(NSE) \begin{cases} \rho \boldsymbol{y}_t + \rho[\boldsymbol{y} \cdot \nabla]\boldsymbol{y} - \mu \Delta \boldsymbol{y} + \nabla p = \rho \boldsymbol{u}, & \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \rho_t + [\boldsymbol{y} \cdot \nabla]\rho = 0, & \rho(0) = \rho_0, \\ \operatorname{div} \boldsymbol{y} = 0 & + B.C. \end{cases}$$

- **Problem:** Not clear if red term is w.l.s.c., and not clear if corresponding Lagrange multiplier to mass equation exists and is a function.

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$$J(\rho, \boldsymbol{u}) = \int_0^T \int_\Omega |\rho(t) - \sigma|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \frac{\beta}{2} \int_0^T \mathcal{H}^1(S_\rho)\,\mathrm{d}t + \frac{\alpha}{2} \int_0^T \int_\Omega |\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

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- **Solution:** Add artificial diffusion to equation and approximate Hausdorff measure ("Mortola-Modica", cf. [Braides, 1998])

Analytical problems and strategy

Minimize

$$J_{\delta}(\rho, \boldsymbol{u}) = \boxed{} + \frac{\beta}{2}\left(\delta \int_{\Omega_T} |\nabla\rho|^2 + \frac{1}{\delta}\int_{\Omega_T} W(\rho)\right) + \boxed{}$$

subject to

$$(NSE_{\varepsilon}) \begin{cases} \rho \boldsymbol{y}_t + \rho[\boldsymbol{y}\cdot\nabla]\boldsymbol{y} - \mu\Delta\boldsymbol{y} + \nabla p = \rho\boldsymbol{u}, & \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \rho_t + [\boldsymbol{y}\cdot\nabla]\rho - \varepsilon\Delta\rho_t = 0, & \rho(0) = \rho_0, \\ \text{div}\, \boldsymbol{y} = 0 & + B.C. \end{cases}$$

($W \geq 0$ double Well functional with $W(\rho) = 0$ iff $\rho = \rho_1$ or $\rho = \rho_2$)

- **Solution:** Add artificial diffusion to equation and approximate Hausdorff measure ("Mortola-Modica", cf. [Braides, 1998])

Analytic results

Theorem (Existence)

For $\delta, \varepsilon > 0$, there exists at least one minimum and the corresponding Lagrange multipliers belong to some $L^p(\Omega_T)$ for $p > 1$.

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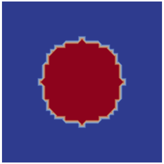
Passing to the limit for $\varepsilon, \delta \to 0$?

Necessary condition for convergence of the whole system is

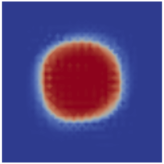
$$\delta \approx \varepsilon.$$

Case $\varepsilon \ll \delta$: parasitic currents

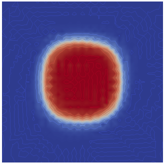
$$\min \delta \int_{\Omega_T} |\nabla \rho|^2 + \frac{1}{\delta} \int_{\Omega_T} W(\rho) \quad \text{s.t.} \quad (NSE_\varepsilon).$$



$\rho(t = 0)$



$\rho(t = 0.25)$



$\rho(t = 0.5)$

Case $\varepsilon \ll \delta$: parasitic currents

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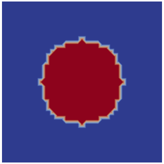

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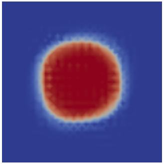

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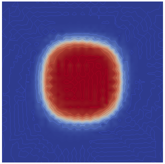

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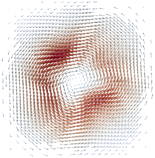
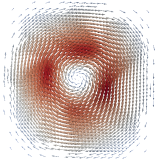
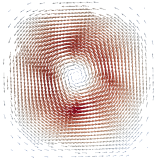


$\boldsymbol{y}(t = 0.05)$



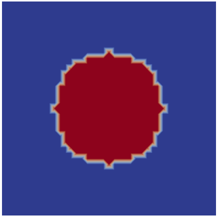
$\boldsymbol{y}(t = 0.15)$



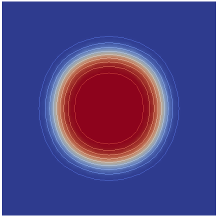
$\boldsymbol{y}(t = 0.35)$

Case $\varepsilon \gg \delta$: massive diffusion

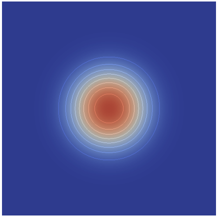
$$\min \; \delta \int_{\Omega_T} |\nabla\rho|^2 + \frac{1}{\delta}\int_{\Omega_T} W(\rho) \quad \text{s.t.} \quad (NSE_\varepsilon).$$



$\rho(t=0)$



$\rho(t=0.5)$
moderate ε



$\rho(t=0.5)$
big ε

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Strategy

Strategy for the discretization

- Fix $\delta, \varepsilon > 0$.
- Use "first discretize, then optimize" ansatz with convergent and unconditionally stable scheme, cf. [Bañas and Prohl, 2010].
- Show existence of discrete optimum, derive discrete optimality conditions.

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- $\Rightarrow$ Bounds for dual variables.

Main result

Theorem (Convergence)

There exist $\boldsymbol{y}, p, \rho; \boldsymbol{z}, q, \eta; \boldsymbol{u} : \Omega_T \to \mathbb{R}^{(2)}$, such that the solutions of the fully discrete optimality system converge to them in some norms (up to subsequences). The limit functions solve the original fully continuous optimality system.

Summary

Done

- New geometric functional considered with PDE constraints: Evidence, existence and optimality conditions for $\delta, \varepsilon > 0$.
- Rigorous converence analysis with unconditionally stable scheme for $\delta, \varepsilon > 0$.
- Implementation for $\delta, \varepsilon > 0$.

Outlook

- What happens for $\varepsilon, \delta \to 0$? Proofs?
- Interplay between δ, ε and numerical parameters (time step size k and grid size h)?
- Surface tension instead of geometric functional?
- Other models (sharp interface, thin film, etc.)?

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Thank you for your attention!

References I

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