HIGHER ORDER TIME DISCRETIZATION FOR THE STOCHASTIC SEMILINEAR WAVE EQUATION WITH MULTIPLICATIVE NOISE

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Abstract. In this paper, a higher order time-discretization scheme is proposed, where the iterates approximate the solution of the stochastic semilinear wave equation driven by multiplicative noise with general drift and diffusion. We employ variational method for its error analysis and prove an improved convergence order of $\frac{3}{2}$ for the approximates of the solution. The core of the analysis is Hölder continuity in time and moment bounds for the solutions of the continuous and the discrete problem. Computational experiments are also presented.

Keywords and phrases: stochastic semilinear wave equations, multiplicative noise, time discretization, stability analysis, convergence rates, strong approximation.

AMS subject classification (2020): 65M12, 65C20, 60H10, 60H15, 60H35, 65C30.

1. Introduction

Let $\mathcal{O} \subset \mathbb{R}^d$, for $1 \le d \le 3$ be a bounded domain. We consider the numerical approximation of the following stochastic semilinear wave equation perturbed by multiplicative noise of Itô type:

$$\begin{cases} \partial_t^2 u + Au = F(u, \partial_t u) + \sigma(u, \partial_t u)\, \partial_t W & \text{in } (0,T) \times \mathcal{O}\,, \\ u(0,\cdot) = u_0\,, \quad \partial_t u(0,\cdot) = v_0 & \text{in } \mathcal{O}\,, \\ u(t,\cdot) = 0 & \text{on } \partial\mathcal{O},\ \forall\, t \in (0,T)\,, \end{cases} \tag{1.1}$$

where A is a strongly elliptic second order differential operator of the form

$$A\varphi(x) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\varphi(x)\Big) \qquad \forall\, x \in \mathcal{O}\,, \tag{1.2}$$

with suitably smooth coefficients $a_{ij}(x)$, where $a^{ij} = a^{ji}\ \forall i,j$, and for every non-zero $\xi \in \mathbb{R}^d$, $\sum_{i,j}^d a_{ij}(x)\xi_i\xi_j \ge \gamma|\xi|^2$, for some constant $\gamma > 0$. Here, F and σ are Lipschitz in both arguments. Let $\mathfrak{P} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, and $\{W(t)\}_{t\ge 0}$ be a finite dimensional Wiener process defined on it; the initial data u_0 and v_0 are given $\mathcal{F}_0-$measurable random variables.

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The work of this author was partially supported by the NSF grant DMS-2012414.
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A strong variational solution to (1.1) exists, see *e.g.* [3, Sec. 6.8], and is usually constructed via the reformulation of $(1.1)_1$ as a first order system by setting $v = \partial_t u$,

$$\begin{cases} \mathrm{d}u = v \,\mathrm{d}t \\ \mathrm{d}v = \big(-Au + F(u,v)\big)\,\mathrm{d}t + \sigma(u,v)\,\mathrm{d}W(t)\,, \end{cases} \tag{1.3}$$

and then using a Faedo-Galerkin method, related uniform energy bounds, and a compactness argument; see Definition 3.1, and Appendix A below. A prototype example is $A = -\Delta$, for which we associate the following energy functional

$$\mathcal{E}(u,v) := \mathcal{E}_{\mathtt{kin}}(v) + \mathcal{E}_{\mathtt{ela}}(u) = \frac{1}{2}\int_{\mathcal{O}} |v(x)|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathcal{O}} |\nabla u(x)|^2 \mathrm{d}x\,, \tag{1.4}$$

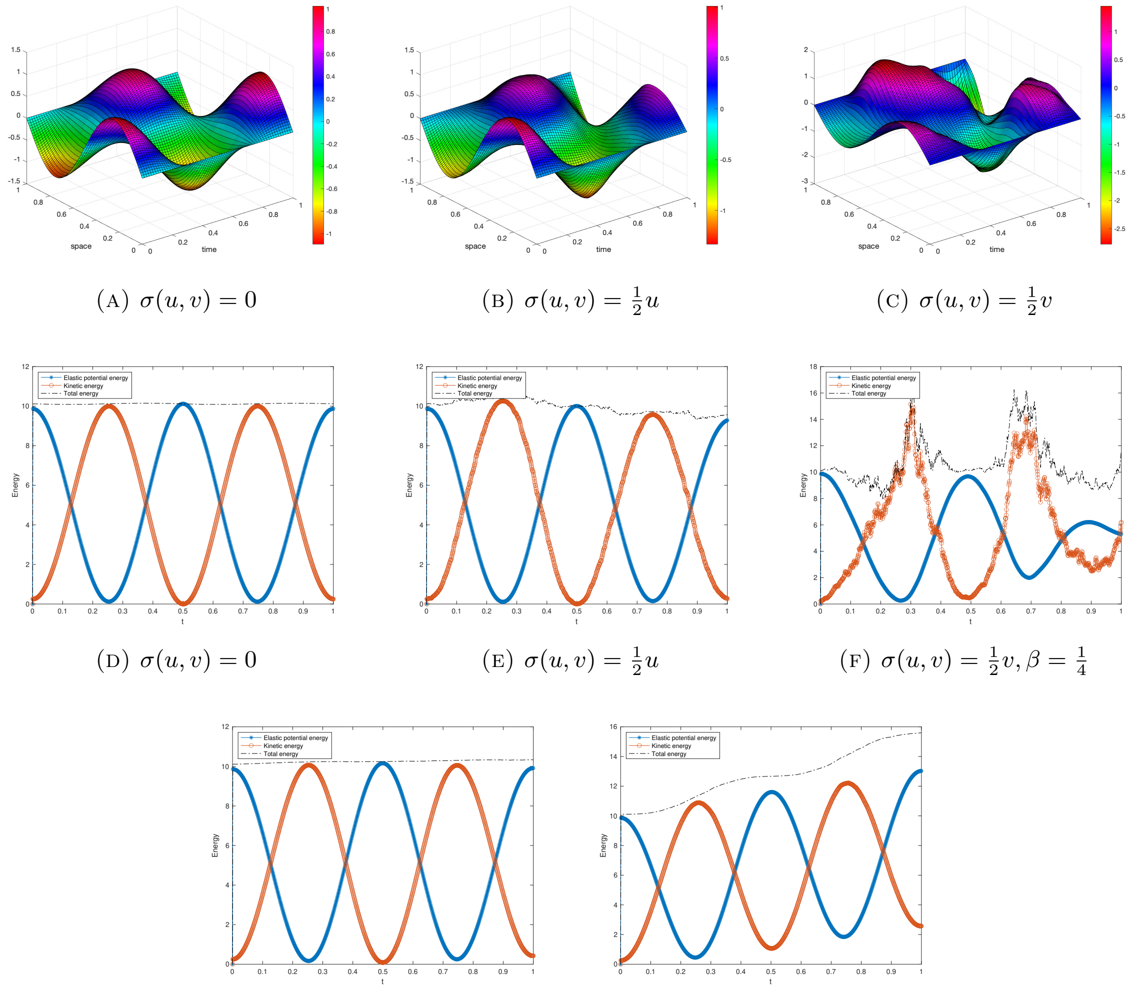
where the first term represents the kinetic energy, and the second the elastic energy of the propagating wave with pointwise elongation $u : [0,T] \times \overline{\mathcal{O}} \times \Omega \to \mathbb{R}$. — We begin the further discussion with an example to motivate the effect of noise.

Example 1. *Let $\mathcal{O} = (0,1)$, $T = 1$, $A = -\Delta$, $F \equiv 0$ in* (1.1)*, and W be of the form* (2.2)*, with $M = 3$, and $e_j(x) = \sqrt{2}\sin(j\pi x)$. — The first line in Fig. 1.1 displays single trajectories of u for different $\sigma \equiv \sigma(u,v)$. For $\sigma \equiv 0$ both, the amplitude and wavelength remain constant over time in snapshot* (A)*, as does $\mathcal{E}(u,v)$ in* (D)*. For $\sigma(u,v) = \frac{1}{2}u$, the amplitude of a single wave realization in snapshot* (B) *changes — as do the trajectory-wise energy parts in* (E) *—, while the wavelength remains constant over time. The computation of the (approximate) total energy uses $\mathtt{MC} = 10^3$ Monte-Carlo simulations in snapshot* (G)*: it is conserved, and close to* (D)*.*

For $\sigma(u,v) = \frac{1}{2}v$ both, the wavelength and frequency of a single trajectory are heavily affected, see snapshot (C)*, and* (F)*, where only $t \mapsto \mathcal{E}_{\mathtt{ela}}(u(t,\omega))$ is smooth. In contrast, the dynamics of $\mathbb{E}_{\mathtt{MC}}[\mathcal{E}(u,v)]$ in* (H) *recovers the exchange of elastic and kinetic energy parts, but the total energy is not conserved any more. The proper resolution of snapshot* (H) *required* 5 *times more Monte-Carlo simulations than for* (G)*.*

The first works to numerically solve (1.1) are [13] and [12], where (semi-)discrete schemes were constructed based on the solution concept of a mild solution for (1.1): in [13], which considered $\mathcal{O} = \mathbb{R}$, $A = -\Delta$, Lipschitz nonlinearities $F \equiv F(u)$ and $\sigma \equiv \sigma(u)$, and white noise, a strong convergence rate $\mathcal{O}(k^{1/2})$ was shown for an explicit finite difference scheme, where the temporal step size k is equal to the mesh size h of the Cartesian spatial mesh; the error analysis uses the Green's function, which is explicitly known in this case, and hence used the mild solution concept for this Cauchy problem.

A further development in this direction is [4], where $\mathcal{O} = (0,1), A = -\Delta$ in (1.1), and the authors used the explicit representation of (discrete) Green's function, such that its implementation crucially hinges on the availability of eigenvalues and eigenfunctions of the Laplacian; see also [3, Sec. 5.3], and [10]. The stable scheme then allows independent choices of k and h, and the proof of [4, Thm. 4.1] provides convergence rates both, in terms of spatial and temporal discretization. We also mention [5], where $\mathcal{O}$ is a bounded convex domain with polygonal boundary, and $A = -\Delta$ in (1.1); the space-time discretization was proposed with the explicit knowledge of the related (discrete) semigroup, whose efficient implementation again hinges on the knowledge of the related eigenvalues and eigenfunctions. Later, in [1] the authors addressed the multiplicative noise case with $\sigma \equiv \sigma(u)$, where σ and also the nonlinearity $F \equiv F(u)$ were assumed to be zero on the boundary. The above mentioned works did not address the case when $F \equiv F(u,v)$ and $\sigma \equiv \sigma(u,v)$.



(A) $\sigma(u, v) = 0$ (B) $\sigma(u, v) = \frac{1}{2}u$ (C) $\sigma(u, v) = \frac{1}{2}v$

(D) $\sigma(u, v) = 0$ (E) $\sigma(u, v) = \frac{1}{2}u$ (F) $\sigma(u, v) = \frac{1}{2}v, \beta = \frac{1}{4}$

(G) $\sigma(u, v) = \frac{1}{2}u$, with $\mathtt{MC} = 10^3$ (H) $\sigma(u, v) = \frac{1}{2}v$, with $\mathtt{MC} = 5 \times 10^3$

FIGURE 1.1. **(Example 1)** (1st line) Single trajectory of u from (1.1), simulated via $(\widehat{\alpha}, \beta)$−scheme ($\widehat{\alpha} = 1$). (2nd line) Corresponding elastic ($\mathcal{E}_{\mathtt{ela}}$), kinetic ($\mathcal{E}_{\mathtt{kin}}$), total energy ($\mathcal{E}$), for mesh sizes $h = 2^{-7}$ and $k = 2^{-10}$. (3rd line) Plots $t \mapsto \mathbb{E}_{\mathtt{MC}}[\mathcal{E}(u(t), v(t))]$, with $\mathtt{MC} = 10^3$ in snapshot (G) and $\mathtt{MC} = 5 \times 10^3$ in (H).

In engineering applications for elastic and acoustic wave propagations which may be described by (1.1), the considered domains $\mathcal{O} \subset \mathbb{R}^d$ are typically complicated, and/or the propagating medium is heterogeneous, with layers, anisotropies, cavities (*e.g.* in seismology, or material testing), or may even be random. Moreover, models of type (1.1) often require non-constant and non-self-adjoint operators, such as those in (1.2), which may even have random coefficients. Therefore, such engineering problems often exclude the efficient use of semigroup based methods through spectral theory as discussed above. This motivates us to aim for the following goals in this work:

1) Use an implicit method in time (below referred to as $(\widehat{\alpha},\beta)-$method, where $\widehat{\alpha}=0$; see Scheme 2) to approximate (1.1) with $F \equiv F(u,\partial_t u)$ and $\sigma \equiv \sigma(u,\partial_t u)$, and employ variational methods for its error analysis. This part is motivated by [7] for the deterministic linear wave equation, *i.e.*, $F \equiv \sigma \equiv 0$. For finite dimensional noise of type (2.2), we use energy arguments to obtain $\mathcal{O}(k)$ for the temporal error — which coincides with the order obtained in [1, Thm. 4] and [4, Thm. 4.1] for an exponential integrator, in the case $\sigma \equiv \sigma(u)$, $F \equiv F(u)$ and trace-class noise in (1.1). We obtain $\mathcal{O}(k^{\frac{1}{2}})$ for the temporal error in the general case $\sigma \equiv \sigma(u,v)$ and $F \equiv F(u,v)$, which has not been addressed in the existing literature.
2) For $\sigma \equiv \sigma(u)$ and $F \equiv F(u)$, in fact, we improve the $(\widehat{\alpha},\beta)-$method to a higher-order method which yields improved convergence order $\mathcal{O}(k^{3/2})$ for approximates of u in $\mathbb{L}^2$; see Theorem 5.1. The additional term that arises for $\widehat{\alpha}=1$ is *motivated* by Ito's formula, and uses increments

$$\widetilde{\Delta_n W} := \int_{t_n}^{t_{n+1}} (s-t_n)\,\mathrm{d}W(s) = \int_{t_n}^{t_{n+1}} s\,\mathrm{d}W(s) - t_n \Delta_n W\,. \tag{1.5}$$

3) Computational experiments in Section 6 show that these results are sharp *w.r.t.* the used noise, *i.e.*, there are examples for $\sigma \equiv \sigma(u,v)$ where the error converges only in order $\mathcal{O}(k)$ — rather than $\mathcal{O}(k^{3/2})$ in the case $\sigma \equiv \sigma(u)$.

In this work, we focus on proper time discretizations for (1.1), which we consider to be the essential part of an overall discretization, and leave a related finite element error analysis for future work. The results will be derived for (1.1) with $A=-\Delta$ to simplify the technical setup, but easily generalize to A in (1.2), even with random coefficients there. Moreover, the $(\widehat{\alpha},\beta)-$method is neither a spectral Galerkin method nor does its implementation hinge on related semigroups.

While being inspired by the second order time-stepping scheme of [7] for the deterministic wave equation, where $u^{n,\frac{1}{2}} := \frac{1}{2}(u^{n+1}+u^{n-1})$, we propose the following scheme for (1.1):

Scheme 1. **($(\widetilde{\alpha},\beta)-$scheme)** *Fix $\widetilde{\alpha} \in \{0,1\}$ and $\beta \in [0,1/2)$. Let $\{t_n\}_{n=0}^N$ be a mesh of size $k>0$ covering $[0,T]$, and $\{(u^n,v^n)_{n=0,1}\}$ be given $\mathcal{F}_{t_n}$-measurable, $[\mathbb{H}_0^1]^2$-valued r.v's. For every $n \geq 1$, find $[\mathbb{H}_0^1]^2$-valued, $\mathcal{F}_{t_{n+1}}$-measurable r.v's (u^{n+1},v^{n+1}) such that $\mathbb{P}$-a.s.*

$$(u^{n+1}-u^n,\phi) = k(v^{n+1},\phi) \quad \forall \phi \in \mathbb{L}^2\,, \tag{1.6}$$

$$\begin{aligned}(v^{n+1}-v^n,\psi) &= -k\big(\nabla \widetilde{u}^{n,\frac{1}{2}},\nabla\psi\big) + \Big(\sigma(u^n,v^{n-\frac{1}{2}})\,\Delta_n W,\psi\Big) \\ &\quad +\widetilde{\alpha}\,\Big(D_u\sigma(u^n,v^{n-\frac{1}{2}})\,v^n\,\widetilde{\Delta_n W},\psi\Big) \\ &\quad +\frac{k}{2}\Big(3F(u^n,v^n) - F(u^{n-1},v^{n-1}),\psi\Big) \quad \forall \psi \in \mathbb{H}_0^1\,,\end{aligned} \tag{1.7}$$

where

$$\widetilde{u}^{n,\frac{1}{2}} \equiv \widetilde{u}_\beta^{n,\frac{1}{2}} := \frac{1+\beta\,k^\beta}{2}u^{n+1} + \frac{1-\beta\,k^\beta}{2}u^{n-1}\,, \tag{1.8}$$

and

$$\Delta_n W := W(t_{n+1}) - W(t_n) \quad \textit{and} \quad v^{n-\frac{1}{2}} := \frac{1}{2}(v^n + v^{n-1}) .$$

Note that $\widetilde{u}^{n,\frac{1}{2}} = u^{n,\frac{1}{2}}$ for $\beta = 0$. Also, in the case when $F \equiv F(u)$, $\sigma \equiv \sigma(u)$ and $\beta = 0$, the $(\widetilde{\alpha}, \beta)-$scheme simplifies to

$$(u^{n+1} - u^n, \phi) = k(v^{n+1}, \phi) \quad \forall \phi \in \mathbb{L}^2 , \tag{1.9}$$

$$\begin{aligned}(v^{n+1} - v^n, \psi) = -k\big(\nabla u^{n,\frac{1}{2}}, \nabla\psi\big) + \Big(\sigma(u^n)\Delta_n W, \psi\Big) + \widetilde{\alpha}\,\Big(D_u\sigma(u^n)v^n\widetilde{\Delta_n W}, \psi\Big) \\ + \frac{k}{2}\Big(3F(u^n) - F(u^{n-1}), \psi\Big) \quad \forall \psi \in \mathbb{H}_0^1 .\end{aligned} \tag{1.10}$$

Scheme 1 involves the increment $\widetilde{\Delta_n W}$ from (1.5). For their implementation, we approximate it by $\widehat{\Delta_n W}$ defined as

$$\widehat{\Delta_n W} := kW(t_{n+1}) - k^2 \sum_{\ell=1}^{k^{-1}} W(t_{n,\ell}) , \tag{1.11}$$

where $\big\{W(t_{n,\ell})\big\}_{\ell=1}^{k^{-1}}$ is the piecewise affine approximation of W on $[t_n, t_{n+1}]$ on an equidistant mesh $\{t_{n,\ell}\}_{\ell=1}^{k^{-1}}$, of size $k^2 := t_{n,\ell+1} - t_{n,\ell}$. To motivate this approximation, we first use Itô's formula to restate $\widetilde{\Delta_n W}$ as

$$\begin{aligned}\widetilde{\Delta_n W} &= \Big(t_{n+1}W(t_{n+1}) - t_n W(t_n)\Big) - \int_{t_n}^{t_{n+1}} W(s)\,\mathrm{d}s - t_n\Delta_n W \\ &= \int_{t_n}^{t_{n+1}} \big[W(t_{n+1}) - W(s)\big]\,\mathrm{d}s = kW(t_{n+1}) - \int_{t_n}^{t_{n+1}} W(s)\,\mathrm{d}s\end{aligned} \tag{1.12}$$

and we approximate the last integral in the right-hand side of (1.12) by $k^2 \sum_{\ell=1}^{k^{-1}} W(t_{n,\ell})$. Thus, we have the following implementable scheme:

Scheme 2. **($(\widehat{\alpha}, \beta)-$scheme)** *Consider Scheme 1. We refer to* (1.6)–(1.7) *as* $(\widehat{\alpha}, \beta)-$*scheme, when* $\widetilde{\alpha}$ *and* $\widetilde{\Delta_n W}$ *are replaced by* $\widehat{\alpha}$ *and* $\widehat{\Delta_n W}$*, respectively.*

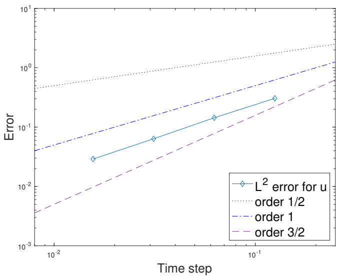
The following example motivates that the convergence rate for the $(1, 0)-$scheme is boosted from $\mathcal{O}(k)$ to $\mathcal{O}(k^{3/2})$, in case $\sigma \equiv \sigma(u)$ and $F \equiv F(u)$.

Example 2. *Let* $\mathcal{O} = (0, 1)$, $T = 1$, $A = -\Delta$, $F \equiv 0$, $\sigma(u) = 2\sin(u)$ *in* (1.3). *Let*

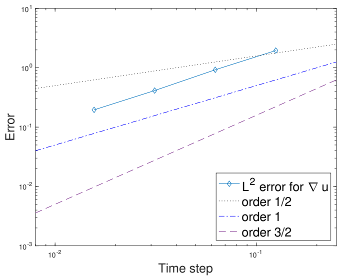
$$u_0(x) = \sin(2\pi x) \qquad \textit{and} \qquad v_0(x) = \sin(3\pi x) ,$$

and W *as in Example 1. Fig. 1.2 displays convergence studies for the scheme* (1.9)–(1.10)*: for* $\widehat{\alpha} = 0$*, the plots* (A) – (C) *show* $\mathbb{L}^2$*-errors in* u, ∇u*, evidencing convergence order* $\mathcal{O}(k)$*, and those for* v *evidence convergence order* $\mathcal{O}(k^{1/2})$*. For* $\widehat{\alpha} = 1$*, the convergence order improves to* $\mathcal{O}(k^{3/2})$ *for* u, ∇u*, and* $\mathcal{O}(k)$ *for* v*; see plots* (D) – (F)*. See Section 6 for more details.*

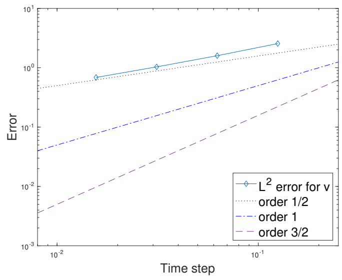
The rest of the paper is organized as follows. In Section 2, we precise the data requirements in (1.3) with $A = -\Delta$ and provide the structure assumptions on F and σ. In Section 3, we recall the concept of a strong variational solution for the problem (1.3) and discuss its regularity. In Section 4, we prove stability results for the $(\widehat{\alpha}, \beta)-$scheme. In Section 5, we prove strong rates of convergence for the above mentioned schemes. In Section 6, we present



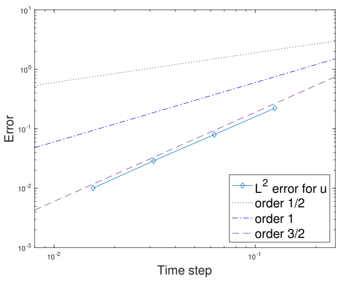
(A) $\mathbb{L}^2$-error for u ($\widehat{\alpha} = 0$ case)



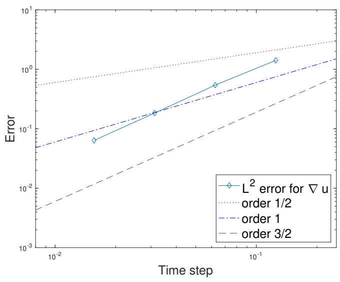
(B) $\mathbb{L}^2$-error for ∇u ($\widehat{\alpha} = 0$ case)



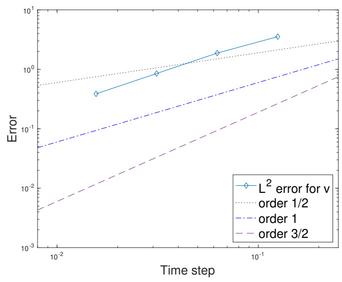
(C) $\mathbb{L}^2$-error for v ($\widehat{\alpha} = 0$ case)



(D) $\mathbb{L}^2$-error for u ($\widehat{\alpha} = 1$ case)



(E) $\mathbb{L}^2$-error for ∇u ($\widehat{\alpha} = 1$ case)



(F) $\mathbb{L}^2$-error for v ($\widehat{\alpha} = 1$ case)

FIGURE 1.2. **(Example 2)** Temporal rates of convergence for the scheme (1.9)–(1.10) with $F \equiv 0$ and $\sigma(u) = 2\sin(u)$; $\widehat{\alpha} = 0$ in (A), (B), (C), and $\widehat{\alpha} = 1$ in (D), (E), (F); discretization parameters: $h = 2^{-7}, k = \{2^{-3}, \cdots, 2^{-6}\}$, $\mathtt{MC} = 3000$.

comparative computational studies which evidence the role of noise in various cases and validates the proved error estimate results.

2. Preliminaries and Assumptions

2.1. **Notation and useful results.** Let $(L^p(\mathcal{O}), \|\cdot\|_{L^p})$ and $(W^{m,p}(\mathcal{O}), \|\cdot\|_{W^{m,p}})$ be the Lebesgue and Sobolev spaces respectively, endowed with usual norms for $m \in \mathbb{N}$ and $1 \le p \le \infty$. We denote $\mathbb{L}^p := L^p(\mathcal{O})$ and $\mathbb{W}^{m,p} := W^{m,p}(\mathcal{O})$. For $p = 2$, let $(\cdot,\cdot)$ be the inner-product in $\mathbb{L}^2$, and $\mathbb{H}^m := \mathbb{W}^{m,2}$. We define $\mathbb{H}^1_0 := \{u \in \mathbb{H}^1 : u|_{\partial\mathcal{O}} = 0\}$.

Let $\mathbb{X}, \mathbb{Y}$ be two separable Hilbert spaces. Let $\mathcal{L}(\mathbb{X},\mathbb{Y})$ denote the space of all linear maps from $\mathbb{X}$ to $\mathbb{Y}$, and $\mathcal{L}_m(\mathbb{X},\mathbb{Y})$ denotes the space of all multi-linear maps from $\mathbb{X} \times \cdots \times \mathbb{X}$ (m-times) to $\mathbb{Y}$ for $m \ge 2$. Throughout this paper, for some $\Phi : \mathbb{H}^1_0 \times \mathbb{H}^1_0 \to \mathbb{L}^2$, we use the notation $D_u\Phi(u,v) \in \mathcal{L}(\mathbb{H}^1_0, \mathbb{L}^2)$ for the Gateaux derivative w.r.t u, whose action is seen as

$$h \mapsto D_u\Phi(u,v)(h), \quad \text{for } h \in \mathbb{H}^1_0 .$$

We denote the second derivative w.r.t. u by $D^2_u\Phi(u,v) \in \mathcal{L}_2(\mathbb{H}^1_0, \mathbb{L}^2)$, whose action can be seen as

$$(h,k) \mapsto D^2_u\Phi(u,v)(h,k) := (D^2_u\Phi(u,v)h)(k) \quad \text{for } (h,k) \in [\mathbb{H}^1_0]^2 .$$

Similarly, we define $D_v\Phi(u,v), D_v^2\Phi(u,v)$.

2.1.1. *A quadrature formula.* The following quadrature formula will be crucially used in our error analysis (see [6, Thm. 2]).

Lemma 2.1. *Let $f \in C^{1,\gamma}([0,T];\mathbb{R})$, for some $\gamma \in (0,1]$. Then there holds*

$$\Big|\frac{f(0)+f(T)}{2} - \frac{1}{T}\int_0^T f(\xi)\,\mathrm{d}\xi\Big| \le \frac{\widetilde{C}}{(\gamma+2)(\gamma+3)}T^{1+\gamma},$$

where $\widetilde{C} > 0$ satisfies

$$\big|f'(t) - f'(s)\big| \le \widetilde{C}|t-s|^\gamma \qquad \forall\, s,t \in [0,T]\,. \tag{2.1}$$

2.2. **Assumptions.** In this section, we list all the assumptions and hypotheses that are imposed throughout this paper.

2.2.1. *Domain and initial data.* We make the following assumptions.

(A1) Let $\mathcal{O} \subset \mathbb{R}^d$, for $1 \le d \le 3$, be a bounded domain
- (i) with $\partial\mathcal{O}$ of class C^1, and $(u_0,v_0) \in \mathbb{H}_0^1 \times \mathbb{L}^2$,
- (ii) with $\partial\mathcal{O}$ of class C^2, and $(u_0,v_0) \in (\mathbb{H}_0^1 \cap \mathbb{H}^2) \times \mathbb{H}_0^1$,
- (iii) with $\partial\mathcal{O}$ of class C^3, and $(u_0,v_0) \in (\mathbb{H}_0^1 \cap \mathbb{H}^3) \times (\mathbb{H}_0^1 \cap \mathbb{H}^2)$.
- (iv) with $\partial\mathcal{O}$ of class C^4, and $(u_0,v_0) \in (\mathbb{H}_0^1 \cap \mathbb{H}^4) \times (\mathbb{H}_0^1 \cap \mathbb{H}^3)$.

2.2.2. *Probability set-up.* For simplicity, let W be a finite-dimensional Wiener process.

(A2) Let $\mathfrak{P} := \big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P}\big)$ be a stochastic basis with a complete filtration $\{\mathcal{F}_t\}_{t\ge0} \subseteq \mathcal{F}$. For some $M \in \mathbb{N}$, let W be a $\mathbb{K}$-valued Wiener process on $\mathfrak{P}$ of the form

$$W(t,x,\omega) := \sum_{j=1}^{M} \beta_j(t,\omega)e_j(x)\,, \tag{2.2}$$

where $\mathbb{K} \subseteq \mathbb{H}_0^1 \cap \mathbb{H}^3$ is a Hilbert space, and $\big\{\beta_j(t,\omega); t \ge 0\big\}$ are mutually independent Brownian motions relative to $\{\mathcal{F}_t\}_{t\ge0}$, and $\{e_j\}_{j=1}^M$ be an orthonormal basis of $\mathbb{K}$.

2.2.3. *The nonlinearity of the model.* Let $F : [\mathbb{H}_0^1]^2 \to \mathbb{L}^2$ and $\sigma : [\mathbb{H}_0^1]^2 \to \mathbb{H}_0^1$.

(A3) Assume $F(u,v) = F_1(u) + F_2(v)$ and $\sigma(u,v) = \sigma_1(u) + \sigma_2(v)$, such that $F_2(v)$ and $\sigma_2(v)$ are affine in v. For any $u, \tilde{u} \in \mathbb{H}_0^1$, there is a constant $C_{\mathrm{L}} > 0$ such that the Lipschitz condition holds:

$$\|F_1(u) - F_1(\tilde{u})\|_{\mathbb{L}^2} + \|\sigma_1(u) - \sigma_1(\tilde{u})\|_{\mathbb{L}^2} \le C_{\mathrm{L}}\,\|u - \tilde{u}\|_{\mathbb{L}^2}\,.$$

(A4) There exists a constant $C_g > 0$ such that

$$\|D_u^m F_1(\cdot)\|_{L^\infty(\mathbb{H}_0^1;\mathcal{L}_m(\mathbb{H}_0^1,\mathbb{L}^2))} + \|D_u^m \sigma_1(\cdot)\|_{L^\infty(\mathbb{H}_0^1;\mathcal{L}_m(\mathbb{H}_0^1,\mathbb{H}_0^1))} \le C_g \quad (m = 1,2,3)\,.$$

By the assumption **(A4)**, we deduce that $F(u,v) \in \mathbb{H}^1$. Since $F(u,v)$ is not assumed to be zero on the boundary, we introduce the following notation.

(A5) Let $\widehat{F}(u,v) := F(u,v) - F(0,0) = F_1(u) + F_2(v) - F(0,0)$, and assume $F(0,0) \in L^2(0,T;\mathbb{H}^m)$ for $m = 1,2,3$.

3. Definition and properties of solution

We recall the concept of a strong variational solution for (1.3) with $A = -\Delta$ and establish stability results in higher spatial norms, and bounds in temporal Hölder norms.

Definition 3.1. *Assume* $\mathbf{(A1)}_i$, $\mathbf{(A2)}$ *and* $\mathbf{(A3)}$. *We call the tuple* (u, v) *a strong variational solution of* (1.3) *with* $A = -\Delta$ *on the interval* $[0, T]$ *if*

(*i*) (u, v) *is an* $\mathbb{H}_0^1 \times \mathbb{L}^2$*-valued,* $\{\mathcal{F}_t\}$*-adapted process*;
(*ii*) $(u, v) \in L^2\big(\Omega; C([0,T]; \mathbb{H}_0^1)\big) \times L^2\big(\Omega; C([0,T]; \mathbb{L}^2)\big)$; *and*

$$(u(t), \phi) = \int_0^t (v, \phi)\,\mathrm{d}s + (u_0, \phi) \quad \forall \phi \in \mathbb{L}^2\,, \tag{3.1}$$

$$(v(t), \psi) = -\int_0^t \Big[(\nabla u, \nabla \psi) + \big(F(u, v), \psi\big)\Big]\mathrm{d}s + \int_0^t \big(\psi, \sigma(u, v)\,\mathrm{d}W(s)\big) + (v_0, \psi) \quad \forall \psi \in \mathbb{H}_0^1\,, \tag{3.2}$$

holds for each $t \in [0, T]$ $\mathbb{P}$*-a.s..*

(*iii*) *There exists a constant* $C > 0$*, depending on* T, C_{L} *and initial data such that there holds* $\mathbb{P}$*-a.s.*

$$\mathbb{E}\Big[\sup_{0 \le t \le T} \mathcal{E}(u(t), v(t))\Big] \le C\,.$$

The existence of a unique strong variational solution was shown in [3, Sec. 6.8, Thm. 8.4].

Lemma 3.2. *Let* (u, v) *be the strong (variational) solution to the problem* (3.1)–(3.2)*. For* $p \in \mathbb{N}$*, there holds* $\mathbb{P}$*-a.s.*

(*i*) *under the hypotheses* $\mathbf{(A1)}_i$, $\mathbf{(A2)}$*, and* $\mathbf{(A3)}$*, the* $\{\mathcal{F}_t\}_{t \ge 0}-$*adapted process* $(u, v) \in L^{2p}\big(\Omega; L^\infty(0, T; \mathbb{H}^1 \times \mathbb{L}^2)\big)$*, and there exists* $K_1 \equiv K_1(p) > 0$*, such that*

$$\mathbb{E}\Big[\sup_{0 \le t \le T} \Big(\|u(t)\|_{\mathbb{H}^1}^{2p} + \|v(t)\|_{\mathbb{L}^2}^{2p}\Big)\Big] \le K_1\,; \tag{3.3}$$

(*ii*) *under the hypotheses* $\mathbf{(A1)}_{ii}$, $\mathbf{(A2)}$, $\mathbf{(A3)}$*, and* $\mathbf{(A4)}$, $\mathbf{(A5)}$ *for* $m = 1$*, the* $\{\mathcal{F}_t\}_{t \ge 0}-$*adapted process* $(u, v) \in L^{2p}\big(\Omega; L^\infty(0, T; \mathbb{H}^2 \times \mathbb{H}^1)\big)$*, and there exists* $K_2 \equiv K_2(p) > 0$*, such that*

$$\mathbb{E}\Big[\sup_{0 \le t \le T} \Big(\|u(t)\|_{\mathbb{H}^2}^{2p} + \|v(t)\|_{\mathbb{H}^1}^{2p}\Big)\Big] \le K_2\,; \tag{3.4}$$

(*iii*) *under the hypotheses* $\mathbf{(A1)}_{iii}$, $\mathbf{(A2)}$, $\mathbf{(A3)}$*, and* $\mathbf{(A4)}$, $\mathbf{(A5)}$ *for* $m = 1, 2$*, the* $\{\mathcal{F}_t\}_{t \ge 0}-$ *adapted process* $(u, v) \in L^{2p}\big(\Omega; L^\infty(0, T; \mathbb{H}^3 \times \mathbb{H}^2)\big)$*, and there exists* $K_3 \equiv K_3(p) > 0$*, such that*

$$\mathbb{E}\Big[\sup_{0 \le t \le T} \Big(\|u(t)\|_{\mathbb{H}^3}^{2p} + \|v(t)\|_{\mathbb{H}^2}^{2p}\Big)\Big] \le K_3\,; \tag{3.5}$$

(*iv*) *under the hypotheses* $\mathbf{(A1)}_{iv}$, $\mathbf{(A2)}$, $\mathbf{(A3)}$*, and* $\mathbf{(A4)}$, $\mathbf{(A5)}$ *for* $m = 1, 2, 3$*, the* $\{\mathcal{F}_t\}_{t \ge 0}-$ *adapted process* $(u, v) \in L^{2p}\big(\Omega; L^\infty(0, T; \mathbb{H}^4 \times \mathbb{H}^3)\big)$*, and there exists* $K_4 \equiv K_4(p) > 0$*, such that*

$$\mathbb{E}\Big[\sup_{0 \le t \le T} \Big(\|u(t)\|_{\mathbb{H}^4}^{2p} + \|v(t)\|_{\mathbb{H}^3}^{2p}\Big)\Big] \le K_4\,. \tag{3.6}$$

Proof. The proof is given in Appendix A. □

3.1. **Hölder continuity in time.** In this subsection, we derive temporal Hölder continuity estimates for the solution pair (u, v) of the problem (3.1)–(3.2) with respect to different norms, which will be useful in the error analysis in later section.

Lemma 3.3. *Let (u, v) be the strong (variational) solution to the problem (3.1)–(3.2). Then, for any $s, t \in [0, T]$, we have for $p \geq 1$*

(i) *under the hypotheses $\mathbf{(A1)}_i$, $\mathbf{(A2)}$, and $\mathbf{(A3)}$, there holds $\mathbb{P}$-a.s.*

$$\Big(\mathbb{E}\Big[\sup_{s\leq r\leq t} \|u(r)-u(s)\|_{\mathbb{L}^2}^{2p}\Big]\Big)^{1/2p} \leq C(K_1)\,|t-s|\,;$$

(ii) *under the hypotheses $\mathbf{(A1)}_{ii}$, $\mathbf{(A2)}$, $\mathbf{(A3)}$, and $\mathbf{(A4)}$, $\mathbf{(A5)}$ for $m = 1$, there holds $\mathbb{P}$-a.s.*

$$\Big(\mathbb{E}\Big[\sup_{s\leq r\leq t} \|u(r)-u(s)\|_{\mathbb{H}^1}^{2p}\Big]\Big)^{1/2p} + \mathbb{E}\Big[\sup_{s\leq r\leq t} \|v(r)-v(s)\|_{\mathbb{L}^2}^{2}\Big] \leq C(K_2)\,|t-s|\,;$$

(iii) *under the hypotheses $\mathbf{(A1)}_{iii}$, $\mathbf{(A2)}$, $\mathbf{(A3)}$, and $\mathbf{(A4)}$, $\mathbf{(A5)}$ for $m = 1, 2$, there holds $\mathbb{P}$-a.s.*

$$\Big(\mathbb{E}\Big[\sup_{s\leq r\leq t} \|u(r)-u(s)\|_{\mathbb{H}^2}^{2p}\Big]\Big)^{1/2p} + \mathbb{E}\Big[\sup_{s\leq r\leq t} \|v(r)-v(s)\|_{\mathbb{H}^1}^{2}\Big] \leq C(K_3)\,|t-s|\,,$$

(iv) *under the hypotheses $\mathbf{(A1)}_{iv}$, $\mathbf{(A2)}$, $\mathbf{(A3)}$, and $\mathbf{(A4)}$, $\mathbf{(A5)}$ for $m = 1, 2, 3$, there holds $\mathbb{P}$-a.s.*

$$\Big(\mathbb{E}\Big[\sup_{s\leq r\leq t} \|u(r)-u(s)\|_{\mathbb{H}^3}^{2p}\Big]\Big)^{1/2p} + \mathbb{E}\Big[\sup_{s\leq r\leq t} \|v(r)-v(s)\|_{\mathbb{H}^2}^{2}\Big] \leq C(K_4)\,|t-s|\,,$$

where the positive constants $C(K_i)$ for $i = 1, \cdots, 4$, depend on the constants K_i, defined in Lemma 3.2.

Proof. The proof is given in Appendix B. ∎

4. Discrete Stability Analysis for the $(\widehat{\alpha}, \beta)$−scheme

If compared to the term $-\Delta u^{n,\frac{1}{2}}$, the term $-\Delta \widetilde{u}^{n,\frac{1}{2}}$ in the $(\widehat{\alpha}, \beta)$−scheme fortifies stability properties of the method: in fact, the identity

$$\widetilde{u}^{n,\frac{1}{2}} = u^{n,\frac{1}{2}} + \beta\frac{k^\beta}{2}\big(u^{n+1} - u^{n-1}\big) = u^{n,\frac{1}{2}} + \beta k^{1+\beta} v^{n+\frac{1}{2}} \tag{4.1}$$

creates an additional 'numerical dissipation' term scaled by $\beta k^{2+\beta}$ in (1.3), which suffices to control general noises $\sigma \equiv \sigma(u, v)$, in case $0 \leq \beta < \frac{1}{2}$ (see Lemma 4.1 below); for $\sigma \equiv \sigma(u)$ only, the scheme (1.9)–(1.10) yields a stable scheme.

In this section, we discuss the discrete stability analysis for the $(\widehat{\alpha}, \beta)$−scheme and we make a remark on the stability results of the scheme (1.9)–(1.10) as this is a sub-case of the $(\widehat{\alpha}, \beta)$−scheme. We recall (1.4), where the energy functional is stated.

(B1) For the stability results, we need the following assumptions on the iterates (u^1, v^1):

(i) Along with $\mathbf{(A1)}_i$, assume $(u^1, v^1) \in L^{2^p}\big(\Omega; [\mathbb{H}_0^1]^2\big)$ for $p \geq 1$.

(ii) Along with $\mathbf{(A1)}_{ii}$, assume $(u^1, v^1) \in L^{2^p}\big(\Omega; [\mathbb{H}_0^1 \cap \mathbb{H}^2]^2\big)$ for $p \geq 1$.

Lemma 4.1. *Let $\widehat{\alpha} \in \{0,1\}$. Assume* $(\mathbf{A1})_{ii}$, **(A2)**, **(A3)**, **(A4)** *for $m = 1$, and* $(\mathbf{B1})_i$. *Then, there exists an $\big[\mathbb{H}_0^1\big]^2$-valued $\{(\mathcal{F}_{t_n})_{0\leq n\leq N}\}$-adapted solution $\{(u^n, v^n);\, 0 \leq n \leq N\}$ of the $(\widehat{\alpha}, \beta)-$scheme. Moreover, for $0 \leq \beta < \frac{1}{2}$, and $k \leq k_0(C_{\mathtt{L}}, C_g)$ sufficiently small, there exists a constant $C_1 > 0$ that does not depend on $k > 0$ such that*

$$\max_{1\leq n\leq N-1} \mathbb{E}\Big[\mathcal{E}(u^{n+1}, v^{n+1})\Big] + \beta k^{2+\beta} \sum_{n=1}^{N-1} \mathbb{E}\Big[\|\nabla v^{n+\frac{1}{2}}\|_{\mathbb{L}^2}^2\Big] \leq C_1 \,. \tag{4.2}$$

In addition, there exists a further constant $C_{2,p} > 0$ such that we have

$$\max_{1\leq n\leq N-1} \mathbb{E}\big[\mathcal{E}^{2^p}(u^{n+1}, v^{n+1})\big] \leq C_{2,p} \qquad (p \geq 1)\,. \tag{4.3}$$

Additionally, assume $\sigma_2(v) \equiv 0 \equiv F_2(v)$ in **(A3)** *and $\beta = 0$. For $k \leq k_0(C_{\mathtt{L}}, C_g)$ sufficiently small, there exists a constant $C_3 > 0$ independent of $k > 0$ such that*

$$\max_{1\leq n\leq N} \mathbb{E}\Big[\|u^n\|_{\mathbb{L}^2}^2\Big] + \frac{1}{4}\mathbb{E}\Big[k\sum_{j=1}^{n}\|\nabla u^j\|_{\mathbb{L}^2}^2\Big] \leq C_3\,. \tag{4.4}$$

There exists further constant $C_{4,p} > 0$ such that we have

$$\max_{1\leq n\leq N} \mathbb{E}\Big[\|u^n\|_{\mathbb{L}^2}^{2^p}\Big] \leq C_{4,p} \qquad (p \geq 1)\,. \tag{4.5}$$

The following remark discusses specific problems to derive this stability result.

Remark 1. 1. *The derivation of (discrete) stability estimates for a (temporal) discretization for the stochastic wave equation* (1.3) *— like Scheme 1 — differs from corresponding tasks for parabolic SPDEs, which is due to the* conservation of energy *in the deterministic case. In this case (where $\sigma \equiv 0$), the test function $v^{n+1/2}$ is 'natural' to deduce* (4.2)*; it is used in* [7] *as well, and exploits the (third) binomial formula, such that the first term in* (1.7) *becomes*

$$\big(v^{n+1} - v^n, \frac{1}{2}[v^{n+1} + v^n]\big) = \frac{1}{2}\big(\|v^{n+1}\|_{\mathbb{L}^2}^2 - \|v^n\|_{\mathbb{L}^2}^2\big)\,;$$

see also (4.7) *below. Conceptional difficulties now appear in the stochastic case where $\sigma \neq 0$ — see* e.g. *the term $\mathfrak{J}_1^n$ in* (4.7)*. A well-known strategy in a setting of parabolic SPDEs would be to employ $\mathbb{E}\big[\Delta_n W\big] = 0$ to conclude*

$$\begin{aligned} \mathfrak{J}_1^n &= \mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac{1}{2}})\,\Delta_n W, \frac{1}{2}[v^{n+1} - v^n]\Big)\Big] \\ &\leq C_\delta\,\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac{1}{2}})\,\Delta_n W\|_{\mathbb{L}^2}^2\Big] + \delta\,\mathbb{E}\big[\|v^{n+1} - v^n\|_{\mathbb{L}^2}^2\big] \qquad (\delta > 0)\,, \end{aligned} \tag{4.6}$$

and to then absorb the last term by a corresponding one on the left-hand side — which would arise if v^{n+1} instead would have been chosen as test function. We avoid the estimation in (4.6) *by using the equation* (1.7) *to replace $v^{n+1} - v^n$ in $\mathfrak{J}_1^n$; see* (4.10) *below.*

2. *The last term in* (4.1) *is the reason to evaluate σ resp. $D_u\sigma$ at $(u^n, v^{n-\frac{1}{2}})$ in* (1.7) *— instead of* e.g. *at (u^n, v^n); see also the left-hand side of* (4.9)*, and the estimation of the terms $\mathfrak{J}_{1,2}^{n,1}$, $\mathfrak{J}_{1,1}^{n,2}$, and $\mathfrak{J}_{1,2}^{n,2}$ in the proof of Lemma 4.1.*

3. *The inequality* (4.3) *assembles higher moment estimates, which will be used in Section 5 to derive improved rates of convergence; see Theorem 5.1.*

4. *The following two inequalities will be used frequently in the proof of discrete stability; see parts* **1a)** *and* **1b)** *below. The identity* (1.12) *leads to*

$$\mathbb{E}\Big[|\widetilde{\Delta_n W}|^2\Big] \leq k\int_{t_n}^{t_{n+1}} \mathbb{E}\Big[|W(t_{n+1})-W(s)|^2\Big]\,\mathrm{d}s \leq Ck^3\,,$$

and by the identity (1.11)*, for* $q=1,2$ *we infer*

$$\mathbb{E}\Big[|\widehat{\Delta_n W}|^{2q}\Big] \leq Ck^{2q}\,\mathbb{E}\big[|W(t_{n+1})|^{2q}\big] + Ck^{2q+1}\sum_{\ell=1}^{k^{-1}}\mathbb{E}\big[|W(t_{n,\ell})|^{2q}\big] \leq Ck^{3q}+Ck^{4q}\leq Ck^{3q}\,.$$

The estimation of the distance between $\widetilde{\Delta_n W}$ *and* $\widehat{\Delta_n W}$ *is useful in the convergence analysis in the next section, which is derived in* (5.9)*.*

5. *If* $\sigma_2(v)\equiv 0\equiv F_2(v)$ *in* **(A3)** *and* $\beta=0$ *hold, we consider the scheme* (1.9)*–*(1.10)*, where to verify discrete stability is easier; in fact, the identity (4.7) simplifies considerably since both,* F *and* σ *do not depend on* v *any more; see* e.g. *the estimate for* $\mathfrak{J}_{1,1}^{n,1}$ *in* (4.12)*. For the higher moment estimates, we multiply the corresponding* (4.19) *of the scheme* (1.9)*–*(1.10) *with* $\mathfrak{E}(u^{n+1},v^{n+1})$ *only, which is different from the general case; see step* **2)** *below.*

6. *In the proof of* (4.4)*, under the hypotheses* $\sigma_2(v)\equiv 0\equiv F_2(v)$ *in* **(A3)***,* $\beta=0$ *and* **(A4)** *for* $m=1$*, we combine both equations of the scheme* (1.9)*–*(1.10) *to write a single equation for* u^ℓ *and sum over the first* n *steps; see* (4.23)*. If we would apply the same approach for the general* $\sigma\equiv\sigma(u^\ell,v^{\ell-1/2})$*, we could use the first equation of the* $(\widehat{\alpha},\beta)$*−scheme to replace* $v^{\ell-1/2}$ *by* $\frac{1}{2k}\big(u^\ell-u^{\ell-2}\big)$*. Thus, to estimate* (4.27) *in general case, we use the growth condition to write*

$$k^2\sum_{\ell=1}^{n}\mathbb{E}\Big[\|\sigma(u^\ell,(1/2k)(u^\ell-u^{\ell-2}))\|_{\mathbb{L}^2}^2\Big] \leq C_{\mathrm{L}}k^2\sum_{\ell=1}^{n}\mathbb{E}\Big[1+\|\nabla u^\ell\|_{\mathbb{L}^2}^2+\frac{1}{4k^2}\big(\|u^\ell\|_{\mathbb{L}^2}^2+\|u^{\ell-2}\|_{\mathbb{L}^2}^2\big)\Big]\,,$$

which, by to the last term, is not in suitable form to apply the discrete Gronwall lemma. Thus, the approach to prove (4.4) *is not useful for the general* $\sigma\equiv\sigma(u,v)$*.*

Proof of Lemma 4.1. The $\mathbb{P}$-a.s. solvability easily follows from Lax-Milgram lemma, and **(A3)**. Using the $\mathbb{L}^2$-regularity theory for elliptic equations on regular domains (see [9, Sec. 15.5]), the system in Scheme 1 holds strongly $\mathrm{Leb}\otimes\mathbb{P}$−a.s. The proof of Lemma 4.1 is split into the following three steps **1)** – **3)**.

1) Proof of (4.2)**.** We use the test function $2kv^{n+1/2}=u^{n+1}-u^{n-1}$ in (1.7), and identity (4.1) to get

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\|v^{n+1}\|_{\mathbb{L}^2}^2-\|v^n\|_{\mathbb{L}^2}^2\Big]+k\mathbb{E}\Big[\big(\nabla u^{n,\frac{1}{2}},\nabla v^{n+\frac{1}{2}}\big)\Big]+\beta\,k^{2+\beta}\mathbb{E}\big[\|\nabla v^{n+\frac{1}{2}}\|_{\mathbb{L}^2}^2\big]\\
&=\mathbb{E}\Big[\Big(\sigma(u^n,v^{n-\frac{1}{2}})\,\Delta_nW,v^{n+\frac{1}{2}}\Big)+\widehat{\alpha}\,\Big(D_u\sigma(u^n,v^{n-\frac{1}{2}})v^n\,\widehat{\Delta_nW},v^{n+\frac{1}{2}}\Big)\\
&\qquad+\frac{k}{2}\Big(3F(u^n,v^n)-F(u^{n-1},v^{n-1}),v^{n+\frac{1}{2}}\Big)\Big]=:\sum_{i=1}^{3}\mathbb{E}\big[\mathfrak{J}_i^n\big]\,.
\end{aligned}\tag{4.7}$$

Next, we use (1.6) in strong form, sum it for two subsequent steps, and multiply this equation with $-\Delta u^{n,\frac{1}{2}}$; we then arrive at

$$\frac{1}{4}\Big[\|\nabla u^{n+1}\|_{\mathbb{L}^2}^2-\|\nabla u^{n-1}\|_{\mathbb{L}^2}^2\Big]=k\Big(\nabla u^{n,\frac{1}{2}},\nabla v^{n+\frac{1}{2}}\Big)\,.\tag{4.8}$$

Since the right-hand side of (4.8) is equal to the second term on the left-hand side of (4.7) we conclude that

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\|v^{n+1}\|_{\mathbb{L}^2}^2 - \|v^n\|_{\mathbb{L}^2}^2\Big] + \frac{1}{4}\mathbb{E}\Big[\|\nabla u^{n+1}\|_{\mathbb{L}^2}^2 - \|\nabla u^{n-1}\|_{\mathbb{L}^2}^2\Big] \\
&\quad + \beta\, k^{2+\beta}\mathbb{E}\big[\|\nabla v^{n+\frac12}\|_{\mathbb{L}^2}^2\big] = \sum_{i=1}^{3}\mathbb{E}\big[\mathfrak{J}_i^n\big]\,.
\end{aligned} \tag{4.9}
$$

Now we estimate each term on the right-hand side of (4.9). Using properties of the increments $\Delta_n W$, we get $\mathbb{E}\big[\big(\sigma(u^n, v^{n-\frac12})\Delta_n W, v^n\big)\big] = 0$. Using this we infer

$$
\begin{aligned}
\mathbb{E}\big[\mathfrak{J}_1^n\big] = \mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, v^{n+\frac12}\Big)\Big] &= \mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, \frac{v^{n+1}+v^n}{2}\Big)\Big] \\
&= \frac{1}{2}\mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, v^{n+1}-v^n\Big)\Big]\,.
\end{aligned}
$$

We use (1.7) — in modified form as stated in Scheme 2 — to replace $v^{n+1}-v^n$. Hence

$$
\begin{aligned}
\mathbb{E}\big[\mathfrak{J}_1^n\big] &= \frac{1}{2}\mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, k\Delta u^{n,\frac12}\Big)\Big] + \frac{1}{2}\mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, \beta k^{2+\beta}\Delta v^{n+\frac12}\Big)\Big] \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac12})\|_{\mathbb{L}^2}^2|\Delta_n W|^2\Big] \\
&\quad + \frac{\widehat{\alpha}}{2}\,\mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, D_u\sigma(u^n, v^{n-\frac12})v^n\,\widehat{\Delta_n W}\Big)\Big] \\
&\quad + \frac{k}{4}\,\mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, \big[3F(u^n, v^n) - F(u^{n-1}, v^{n-1})\big]\Big)\Big] =: \sum_{i=1}^{5}\mathfrak{J}_1^{n,i}\,.
\end{aligned} \tag{4.10}
$$

In the following parts **a)**–**c)**, we independently bound $\mathbb{E}\big[\mathfrak{J}_1^n\big]$ through $\mathbb{E}\big[\mathfrak{J}_3^n\big]$ in (4.7).

a) Estimation of $\mathbb{E}\big[\mathfrak{J}_1^n\big]$ in (4.10). We estimate the five terms $\mathfrak{J}_1^{n,i}$, $i = 1, \cdots, 5$, on the right-hand side of (4.10). Let $\underline{D_u\sigma} \equiv D_u\sigma(u^n, v^{n-\frac12}) \in \mathcal{L}(\mathbb{H}_0^1, \mathbb{H}_0^1)$ and $\underline{D_v\sigma} \equiv D_v\sigma(u^n, v^{n-\frac12}) \in \mathcal{L}(\mathbb{H}_0^1, \mathbb{H}_0^1)$. By integration by parts and using $\sigma(u^n, v^{n-\frac12}) = 0$ on $\partial\mathcal{O}$ we infer

$$
\begin{aligned}
\mathfrak{J}_1^{n,1} &= \frac{1}{2}\mathbb{E}\Big[-\Big(\underline{D_u\sigma}\,\nabla u^n\Delta_n W, k\nabla u^{n,\frac12}\Big)\Big] + \frac{1}{2}\mathbb{E}\Big[-\Big(\underline{D_v\sigma}\,\nabla v^{n-\frac12}\Delta_n W, k\nabla u^{n,\frac12}\Big)\Big] \\
&= \mathfrak{J}_{1,1}^{n,1} + \mathfrak{J}_{1,2}^{n,1}\,.
\end{aligned} \tag{4.11}
$$

Using **(A4)** for $m = 1$ and the Itô isometry we get

$$
\begin{aligned}
\mathfrak{J}_{1,1}^{n,1} &\leq C_g^2\,\mathbb{E}\Big[\|\nabla u^n\|_{\mathbb{L}^2}^2|\Delta_n W|^2\Big] + Ck^2\,\mathbb{E}\Big[\|\nabla u^{n,\frac12}\|_{\mathbb{L}^2}^2\Big] \\
&\leq C_g^2\,k\,\mathbb{E}\big[\|\nabla u^n\|_{\mathbb{L}^2}^2\big] + Ck^2\,\mathbb{E}\Big[\|\nabla u^{n+1}\|_{\mathbb{L}^2}^2 + \|\nabla u^{n-1}\|_{\mathbb{L}^2}^2\Big]\,.
\end{aligned} \tag{4.12}
$$

Using **(A4)** for $m = 1$, the independence property of the increment $\Delta_n W$, the Itô isometry and the identity $2kv^{n+1/2} = u^{n+1} - u^{n-1}$, we estimate

$$
\begin{aligned}
\mathfrak{J}_{1,2}^{n,1} &= \frac{1}{2}\mathbb{E}\Big[-\Big(\underline{D_v\sigma}\,\nabla v^{n-\frac12}\Delta_n W, \frac{k}{2}\,\nabla\big[u^{n+1} - u^{n-1}\big]\Big)\Big] \\
&= \frac{1}{2}\mathbb{E}\Big[-\Big(k^{1-\frac{\beta}{2}}\,\underline{D_v\sigma}\,\nabla v^{n-\frac12}\Delta_n W, k^{1+\frac{\beta}{2}}\,\nabla v^{n+\frac12}\Big)\Big] \\
&\leq \frac{3}{8}\,C_g^2\,k^{1+(2-\beta)}\,\mathbb{E}\Big[\|\nabla v^{n-\frac12}\|_{\mathbb{L}^2}^2\Big] + \frac{1}{6}\,k^{2+\beta}\,\mathbb{E}\Big[\|\nabla v^{n+\frac12}\|_{\mathbb{L}^2}^2\Big]\,.
\end{aligned} \tag{4.13}
$$

The terms on the right-hand sides of (4.12) and (4.13) may now be controlled by those on the left-hand side of (4.9) after summation over $1 \leq n \leq N-1$, provided that $k \leq \big(\frac{4}{3C_g^2 \beta}\big)^{\frac{1}{1-2\beta}}$ for $\beta < \frac{1}{2}$ is sufficiently small.

Now we turn to $\mathfrak{J}_1^{n,2}$ in (4.10): integration by parts and using the fact that $\sigma(u^n, v^{n-\frac{1}{2}}) = 0$ on $\partial\mathcal{O}$ we get

$$\begin{aligned}\mathfrak{J}_1^{n,2} &= \frac{1}{2}\,\mathbb{E}\Big[-\Big(k^{1+\frac{\beta}{2}}\underline{D_u\sigma}\,\nabla u^n\,\Delta_n W, \beta\, k^{1+\frac{\beta}{2}}\,\nabla v^{n+\frac{1}{2}}\Big)\Big] \\ &\quad + \frac{1}{2}\,\mathbb{E}\Big[-\Big(k^{1+\frac{\beta}{2}}\underline{D_v\sigma}\,\nabla v^{n-\frac{1}{2}}\Delta_n W, \beta\, k^{1+\frac{\beta}{2}}\,\nabla v^{n+\frac{1}{2}}\Big)\Big] =: \mathfrak{J}_{1,1}^{n,2} + \mathfrak{J}_{1,2}^{n,2}\,. \end{aligned} \tag{4.14}$$

Using **(A4)** for $m = 1$ and the independence property of $\Delta_n W$ we estimate

$$\mathfrak{J}_{1,1}^{n,2} \leq C\,k^{3+\beta}\,\mathbb{E}\big[\|\nabla u^n\|_{\mathbb{L}^2}^2\big] + \beta\frac{1}{6}\,k^{2+\beta}\,\mathbb{E}\Big[\|\nabla v^{n+\frac{1}{2}}\|_{\mathbb{L}^2}^2\Big]\,,$$

where the second term in the right-hand side can be be controlled by the corresponding term on the left-hand side of (4.9). Again, using **(A4)** for $m = 1$ we obtain for the second term in (4.14)

$$\mathfrak{J}_{1,2}^{n,2} \leq \beta\frac{3}{8}C_g^2\,k\,k^{2+\beta}\,\mathbb{E}\big[\|\nabla v^{n-\frac{1}{2}}\|_{\mathbb{L}^2}^2\big] + \beta\frac{1}{6}\,k^{2+\beta}\,\mathbb{E}\Big[\|\nabla v^{n+\frac{1}{2}}\|_{\mathbb{L}^2}^2\Big]\,,$$

where the right-hand side can be managed with the left-hand side of (4.9).

We continue with the next term $\mathfrak{J}_1^{n,3}$ in (4.10): by Itô isometry and **(A3)**,

$$\mathfrak{J}_1^{n,3} = \frac{1}{2}\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac{1}{2}})\|_{\mathbb{L}^2}^2|\Delta_n W|^2\Big] \leq Ck\,\mathbb{E}\Big[1 + \|\nabla u^n\|_{\mathbb{L}^2}^2 + \|v^n\|_{\mathbb{L}^2}^2 + \|v^{n-1}\|_{\mathbb{L}^2}^2\Big]\,,$$

where $C > 0$ depends on C_{L}. Next comes $\mathfrak{J}_1^{n,4}$: using **(A3)**, **(A4)** for $m = 1$ and item **4.** of Remark 1, we infer

$$\begin{aligned}\mathfrak{J}_1^{n,4} &\leq C\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac{1}{2}})\|_{\mathbb{L}^2}^2|\Delta_n W|^2\Big] + \frac{\widehat{\alpha}^2}{4}C_g^2\,\mathbb{E}\Big[\|v^n\|_{\mathbb{L}^2}^2|\widehat{\Delta_n W}|^2\Big] \\ &\leq Ck\,\mathbb{E}\Big[1 + \|\nabla u^n\|_{\mathbb{L}^2}^2 + \|v^n\|_{\mathbb{L}^2}^2 + \|v^{n-1}\|_{\mathbb{L}^2}^2\Big] + \frac{\widehat{\alpha}^2}{4}C_g^2\,k^3\,\mathbb{E}\big[\|v^n\|_{\mathbb{L}^2}^2\big]\,,\end{aligned} \tag{4.15}$$

where $C > 0$ depends on C_{L}. The last term is $\mathfrak{J}_1^{n,5}$: by **(A3)** we obtain

$$\begin{aligned}\mathfrak{J}_1^{n,5} &\leq C\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac{1}{2}})\|_{\mathbb{L}^2}^2|\Delta_n W|^2\Big] + Ck^2\,\mathbb{E}\Big[\|F(u^n, v^n)\|_{\mathbb{L}^2}^2 + \|F(u^{n-1}, v^{n-1})\|_{\mathbb{L}^2}^2\Big] \\ &\leq Ck\,\mathbb{E}\Big[1 + \|\nabla u^n\|_{\mathbb{L}^2}^2 + \|\nabla u^{n-1}\|_{\mathbb{L}^2}^2 + \|v^n\|_{\mathbb{L}^2}^2 + \|v^{n-1}\|_{\mathbb{L}^2}^2\Big]\,,\end{aligned} \tag{4.16}$$

where the constant $C > 0$ depends on C_{L}. Thus, the estimate of $\mathfrak{J}_1^n$ through those of $\mathfrak{J}_1^{n,1}$ through $\mathfrak{J}_1^{n,5}$ is complete.

b) Estimation of $\mathbb{E}\big[\mathfrak{J}_2^n\big]$ **in (4.7).** By **(A4)** for $m = 1$, item **4.** of Remark 1, and the independence property of $\widehat{\Delta_n W}$,

$$\begin{aligned}\mathfrak{J}_2^n &\leq \widehat{\alpha}^2\frac{1}{k}\,\mathbb{E}\Big[\|D_u\sigma(u^n, v^{n-\frac{1}{2}})v^n\|_{\mathbb{L}^2}^2|\widehat{\Delta_n W}|^2\Big] + Ck\,\mathbb{E}\Big[\|v^{n+\frac{1}{2}}\|_{\mathbb{L}^2}^2\Big] \\ &\leq C_g^2\,\widehat{\alpha}^2\,k^2\,\mathbb{E}\big[\|v^n\|_{\mathbb{L}^2}^2\big] + Ck\,\mathbb{E}\Big[\|v^{n+1}\|_{\mathbb{L}^2}^2 + \|v^n\|_{\mathbb{L}^2}^2\Big]\,.\end{aligned}$$

c) Estimation of $\mathbb{E}\big[\mathcal{J}_3^n\big]$ in (4.7). By **(A3)** we estimate

$$
\begin{aligned}
\mathcal{J}_3^n &= k\mathbb{E}\Big[\Big(F(u^n,v^n),v^{n+\frac12}\Big)\Big] + \frac{k}{2}\,\mathbb{E}\Big[\Big(F(u^n,v^n)-F(u^{n-1},v^{n-1}),v^{n+\frac12}\Big)\Big] \\
&\le Ck\,\mathbb{E}\big[\|v^{n+\frac12}\|_{\mathbb{L}^2}^2\big] + C\,k\,\mathbb{E}\Big[1+\|\nabla u^n\|_{\mathbb{L}^2}^2+\|\nabla u^{n-1}\|_{\mathbb{L}^2}^2+\|v^n\|_{\mathbb{L}^2}^2+\|v^{n-1}\|_{\mathbb{L}^2}^2\Big]\,.
\end{aligned}
\tag{4.17}
$$

Now, we may use the parts **a)** through **c)** to bound the terms on the right-hand side of (4.9). Summation over all $1\le n\le N-1$, for $k\le k_0\equiv k_0(C_{\mathtt{L}},C_g)$ sufficiently small, leads to

$$
\begin{aligned}
&\frac14\mathbb{E}\Big[\mathcal{E}(u^N,v^N)\Big] + \beta\frac14\,k^{2+\beta}\,\mathbb{E}\big[\|\nabla v^{n+\frac12}\|_{\mathbb{L}^2}^2\big] \\
&\le \frac14\mathbb{E}\big[\mathcal{E}(u_0,v^1)\big] + \beta\frac{k^{2+\beta}}{4}\mathbb{E}\big[\|\nabla v^{1/2}\|_{\mathbb{L}^2}^2\big] + \frac14\mathbb{E}\big[\|\nabla u^1\|_{\mathbb{L}^2}^2\big] + Ck\sum_{n=1}^{N-1}\mathbb{E}\big[\mathcal{E}(u^n,v^n)\big]\,.
\end{aligned}
\tag{4.18}
$$

By $\mathbf{(B1)}_i$, the implicit version of the discrete Gronwall lemma then shows (4.2).

2) Proof of (4.3) **for** $p=1$**.** To simplify technicalities, we put $F\equiv 0$. Let us denote $\mathfrak{E}(u^{n+1},v^{n+1}):=\frac14\big[\|\nabla u^{n+1}\|_{\mathbb{L}^2}^2+2\|v^{n+1}\|_{\mathbb{L}^2}^2\big]$. Arguing as before (4.9) then leads to

$$
\begin{aligned}
&\big[\mathfrak{E}(u^{n+1},v^{n+1})-\mathfrak{E}(u^{n-1},v^n)\big] + \beta k^{2+\beta}\|\nabla v^{n+\frac12}\|_{\mathbb{L}^2}^2 \\
&\quad = \Big(\sigma(u^n,v^{n-\frac12})\Delta_n W, v^{n+\frac12}\Big) + \widehat{\alpha}\,\Big(D_u\sigma(u^n,v^{n-\frac12})v^n\,\widehat{\Delta_n W},v^{n+\frac12}\Big)\,.
\end{aligned}
\tag{4.19}
$$

Now fix $\frac14\le\delta_1,\delta_2\le 1$, then multiply (4.19) with

$$
\delta_1\mathfrak{E}(u^{n+1},v^{n+1}) + \delta_2\Big(\mathfrak{E}(u^{n+1},v^{n+1})+\mathfrak{E}(u^{n-1},v^n)\Big)\,,
$$

and take the expectation to get

$$
\begin{aligned}
&\frac{\delta_1+2\delta_2}{2}\mathbb{E}\Big[\mathfrak{E}^2(u^{n+1},v^{n+1})-\mathfrak{E}^2(u^{n-1},v^n)\Big] + \frac{\delta_1}{2}\mathbb{E}\Big[\big|\mathfrak{E}(u^{n+1},v^{n+1})-\mathfrak{E}(u^{n-1},v^n)\big|^2\Big] \\
&\quad + \beta k^{2+\beta}\mathbb{E}\Big[\|\nabla v^{n+\frac12}\|_{\mathbb{L}^2}^2\big[(\delta_1+\delta_2)\mathfrak{E}(u^{n+1},v^{n+1})+\delta_2\mathfrak{E}(u^{n-1},v^n)\big]\Big] \\
&= \mathbb{E}\Big[\Big(\sigma(u^n,v^{n-\frac12})\Delta_n W,v^{n+\frac12}\Big)\cdot\big[(\delta_1+\delta_2)\mathfrak{E}(u^{n+1},v^{n+1})+\delta_2\mathfrak{E}(u^{n-1},v^n)\big]\Big] \\
&\quad + \widehat{\alpha}\,\mathbb{E}\Big[\Big(D_u\sigma(u^n,v^{n-\frac12})v^n\,\widehat{\Delta_n W},v^{n+\frac12}\Big)\cdot\big[(\delta_1+\delta_2)\mathfrak{E}(u^{n+1},v^{n+1})+\delta_2\mathfrak{E}(u^{n-1},v^n)\big]\Big] \\
&=: \mathcal{K}^{n,1}+\mathcal{K}^{n,2}\,.
\end{aligned}
\tag{4.20}
$$

We independently estimate the terms $\mathcal{K}^{n,1}$ and $\mathcal{K}^{n,2}$.

a) Estimation of $\mathcal{K}^{n,1}$ in (4.20). This term may be written as the sum of two others:

$$
\begin{aligned}
\mathcal{K}^{n,1} &= (\delta_1+\delta_2)\mathbb{E}\Big[\Big(\sigma(u^n,v^{n-\frac12})\Delta_n W,v^{n+\frac12}\Big)\cdot\big(\mathfrak{E}(u^{n+1},v^{n+1})-\mathfrak{E}(u^{n-1},v^n)\big)\Big] \\
&\quad + (\delta_1+2\delta_2)\mathbb{E}\Big[\Big(\sigma(u^n,v^{n-\frac12})\Delta_n W,v^{n+\frac12}\Big)\cdot\mathfrak{E}(u^{n-1},v^n)\Big] := \mathcal{K}_1^{n,1}+\mathcal{K}_2^{n,1}\,.
\end{aligned}
\tag{4.21}
$$

We consider $\mathcal{K}_1^{n,1}$ first. By $\mathbb{E}\big[|\Delta_n W|^4\big]=\mathcal{O}(k^2)$, and **(A3)** we find

$$
\begin{aligned}
\mathcal{K}_1^{n,1} &\le C_{\delta_1}\,\mathbb{E}\Big[\|\sigma(u^n,v^{n-\frac12})\Delta_n W\|_{\mathbb{L}^2}^2\|v^{n+\frac12}\|_{\mathbb{L}^2}^2\Big] + \frac{\delta_1}{4}\mathbb{E}\Big[\big|\mathfrak{E}(u^{n+1},v^{n+1})-\mathfrak{E}(u^{n-1},v^n)\big|^2\Big] \\
&\le \frac{C_{\delta_1}}{k}\mathbb{E}\Big[\|\sigma(u^n,v^{n-\frac12})\|_{\mathbb{L}^2}^4|\Delta_n W|^4\Big] + C_{\delta_1}k\,\mathbb{E}\Big[\|v^{n+\frac12}\|_{\mathbb{L}^2}^4\Big] + \frac{\delta_1}{4}\mathbb{E}\Big[\big|\mathfrak{E}(u^{n+1},v^{n+1})-\mathfrak{E}(u^{n-1},v^n)\big|^2\Big] \\
&\le C_{\delta_1}k\,\mathbb{E}\Big[1+\sum_{\ell=-1}^{1}\mathfrak{E}^2(u^{n+\ell},v^{n+\ell})\Big] + \frac{\delta_1}{4}\mathbb{E}\Big[\big|\mathfrak{E}(u^{n+1},v^{n+1})-\mathfrak{E}(u^{n-1},v^n)\big|^2\Big]\,,
\end{aligned}
$$

where the last term on the right-hand side can be absorbed on the left-hand side of (4.20). We continue with $\mathcal{K}_2^{n,1}$: on using ithe ndependence property of $\Delta_n W$, and equation (1.7),

$$\begin{aligned}\mathcal{K}_2^{n,1} &\leq 3\Big|\mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, v^{n+1}-v^n\Big)\cdot \mathfrak{E}(u^{n-1},v^n)\Big]\Big| \\ &= 3\Big|\mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, k\Delta\widetilde{u}^{n,\frac12}\Big)\cdot \mathfrak{E}(u^{n-1},v^n)\Big]\Big| + 3\Big|\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac12})\Delta_n W\|^2_{\mathbb{L}^2}\,\mathfrak{E}(u^{n-1},v^n)\Big]\Big| \\ &\quad + 3\Big|\mathbb{E}\Big[\Big(\sigma(u^n, v^{n-\frac12})\Delta_n W, D_u\sigma(u^n, v^{n-\frac12})v^n\widehat{\Delta_n W}\Big)\cdot \mathfrak{E}(u^{n-1},v^n)\Big]\Big| =: \mathcal{K}_{2,1}^{n,1}+\mathcal{K}_{2,2}^{n,1}+\mathcal{K}_{2,3}^{n,1}\,.\end{aligned}$$

We split $\mathcal{K}_{2,1}^{n,1} := \mathcal{K}_{2,1}^{n,1,\mathtt{A}}+\mathcal{K}_{2,1}^{n,1,\mathtt{B}}$ because of (4.1); here, $\mathcal{K}_{2,1}^{n,1,\mathtt{A}}$ is as $\mathcal{K}_{2,1}^{n,1}$, where $\widetilde{u}^{n,\frac12}$ is replaced by $u^{n,\frac12}$. We use integration by parts, and the fact that $\sigma(u^n, v^{n-\frac12}) = 0$ on $\partial\mathcal{O}$, **(A4)** for $m = 1$, the independence property of $\Delta_n W$ and that $\mathbb{E}\big[|\Delta_n W|^4\big] = \mathcal{O}(k^2)$ to conclude

$$\begin{aligned}\mathcal{K}_{2,1}^{n,1,\mathtt{A}} &= \frac92\Big|\mathbb{E}\Big[-\Big(\nabla\sigma(u^n, v^{n-\frac12})\Delta_n W, k\nabla[u^{n+1}-u^{n-1}]\Big)\cdot \mathfrak{E}(u^{n-1},v^n)\Big]\Big| \\ &= 9\Big|\mathbb{E}\Big[-\Big(\nabla\sigma(u^n, v^{n-\frac12})\Delta_n W, k^{1-\frac{\beta}{2}}k^{1+\frac{\beta}{2}}\,\nabla v^{n+\frac12}\Big)\cdot \mathfrak{E}(u^{n-1},v^n)\Big]\Big| \\ &\leq C_{\delta_2}k^{3-\beta}\,\mathbb{E}\Big[\|\underline{D_u\sigma}\,\nabla u^n\|^2_{\mathbb{L}^2}\mathfrak{E}(u^{n-1},v^n)\Big] + C_{\delta_2}k^{3-\beta}\,\mathbb{E}\Big[\|\underline{D_v\sigma}\,\nabla v^{n-\frac12}\|^2_{\mathbb{L}^2}\mathfrak{E}(u^{n-1},v^n)\Big] \\ &\quad + \frac{\delta_2}{4}k^{2+\beta}\,\mathbb{E}\Big[\|\nabla v^{n+\frac12}\|^2_{\mathbb{L}^2}\,\mathfrak{E}(u^{n-1},v^n)\Big] \\ &\leq C_{\delta_2}C_g^2k^{3-\beta}\mathbb{E}\Big[\mathfrak{E}^2(u^n,v^n)+\mathfrak{E}^2(u^{n-1},v^n)\Big] + C_{\delta_2}C_g^2\,k^{3-\beta}\,\mathbb{E}\Big[\|\nabla v^{n-\frac12}\|^2_{\mathbb{L}^2}\,\mathfrak{E}(u^{n-1},v^n)\Big] \\ &\quad + \frac{\delta_2}{4}k^{2+\beta}\,\mathbb{E}\Big[\|\nabla v^{n+\frac12}\|^2_{\mathbb{L}^2}\,\mathfrak{E}(u^{n-1},v^n)\Big]\,,\end{aligned}$$

and we use a similar idea to estimate $\mathcal{K}_{2,1}^{n,1,\mathtt{B}}$,

$$\begin{aligned}\mathcal{K}_{2,1}^{n,1,\mathtt{B}} &= \frac92\Big|\mathbb{E}\Big[-\Big(\nabla\sigma(u^n, v^{n-\frac12})\Delta_n W, \beta k^{2+\beta}\nabla v^{n+\frac12}\Big)\cdot \mathfrak{E}(u^{n-1},v^n))\Big]\Big| \\ &\leq C_{\delta_2}k^{3+\beta}\,\mathbb{E}\Big[\|\underline{D_u\sigma}\,\nabla u^n\|^2_{\mathbb{L}^2}\mathfrak{E}(u^{n-1},v^n)\Big] + C_{\delta_2}k^{3+\beta}\,\mathbb{E}\Big[\|\underline{D_v\sigma}\,\nabla v^{n-\frac12}\|^2_{\mathbb{L}^2}\mathfrak{E}(u^{n-1},v^n)\Big] \\ &\quad + \beta\frac{\delta_2}{4}k^{2+\beta}\,\mathbb{E}\Big[\|\nabla v^{n+\frac12}\|^2_{\mathbb{L}^2}\,\mathfrak{E}(u^{n-1},v^n)\Big] \\ &\leq C_{\delta_2}C_g^2\,k^{3+\beta}\,\mathbb{E}\Big[\mathfrak{E}^2(u^n,v^n)+\mathfrak{E}^2(u^{n-1},v^n)\Big] + C_{\delta_2}C_g^2\,k^{3+\beta}\,\mathbb{E}\Big[\|\nabla v^{n-\frac12}\|^2_{\mathbb{L}^2}\mathfrak{E}(u^{n-1},v^n)\Big] \\ &\quad + \beta\frac{\delta_2}{4}k^{2+\beta}\,\mathbb{E}\Big[\|\nabla v^{n+\frac12}\|^2_{\mathbb{L}^2}\,\mathfrak{E}(u^{n-1},v^n)\Big]\,,\end{aligned}$$

where the last two terms in the right-hand sides of $\mathcal{K}_{2,1}^{n,1,\mathtt{A}}$ and $\mathcal{K}_{2,1}^{n,1,\mathtt{B}}$ may be controlled by those on the left-hand side of (4.20) after summation over $1 \leq n \leq N-1$, provided that k is sufficiently small and $\beta < \frac12$.
Similarly, using **(A3)**, and $\mathbb{E}\big[|\Delta_n W|^4\big] = \mathcal{O}(k^2)$ we estimate

$$\mathcal{K}_{2,2}^{n,1} \leq \frac{C}{k}\,\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac12})\|^4_{\mathbb{L}^2}|\Delta_n W|^4\Big]+Ck\,\mathbb{E}\Big[\mathfrak{E}^2(u^{n-1},v^n)\Big] \leq Ck\,\mathbb{E}\Big[1+\sum_{\ell=-1}^{0}\mathfrak{E}^2(u^{n+\ell},v^{n+\ell})\Big]\,.$$

Using $\mathbb{E}\big[|\widehat{\Delta_n W}|^4\big] = \mathcal{O}(k^6)$, and **(A3)** gives

$$\begin{aligned}
\mathcal{K}^{n,1}_{2,3} &\leq \frac{C}{k}\,\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac12})\Delta_n W\|^2_{\mathbb{L}^2}\,\|D_u\sigma(u^n, v^{n-\frac12})v^n\widehat{\Delta_n W}\|^2_{\mathbb{L}^2}\Big] + Ck\,\mathbb{E}\Big[\mathfrak{E}^2(u^{n-1}, v^n)\Big] \\
&\leq \frac{C}{k}\,\mathbb{E}\Big[\|\sigma(u^n, v^{n-\frac12})\|^4_{\mathbb{L}^2}|\Delta_n W|^4\Big] + \frac{C_g^4}{k}\mathbb{E}\Big[\|v^n\|^4_{\mathbb{L}^2}\big|\widehat{\Delta_n W}\big|^4\Big] + Ck\,\mathbb{E}\Big[\mathfrak{E}^2(u^{n-1}, v^n)\Big] \\
&\leq Ck\,\mathbb{E}\Big[1 + \sum_{\ell=-1}^{0} \mathfrak{E}^2(u^{n+\ell}, v^{n+\ell})\Big]\,.
\end{aligned}$$

b) Estimation of $\mathcal{K}^{n,2}$ in (4.20). By **(A4)** for $m = 1$ and using the fact that $\mathbb{E}\big[|\widehat{\Delta_n W}|^4\big] = \mathcal{O}(k^6)$, we infer

$$\begin{aligned}
\mathcal{K}^{n,2} &\leq \frac{\widehat{\alpha}^2}{k}\mathbb{E}\Big[\|D_u\sigma(u^n, v^{n-\frac12})v^n\,\widehat{\Delta_n W}\|^2_{\mathbb{L}^2}\,\|v^{n+\frac12}\|^2_{\mathbb{L}^2}\Big] \\
&\quad + Ck\,\mathbb{E}\Big[\mathfrak{E}^2(u^{n+1}, v^{n+1})\Big] + Ck\,\mathbb{E}\Big[\mathfrak{E}^2(u^{n-1}, v^n)\Big] \\
&\leq \frac{\widehat{\alpha}^4}{k^3}\mathbb{E}\Big[\|D_u\sigma(u^n, v^{n-\frac12})v^n\|^4_{\mathbb{L}^2}\big|\widehat{\Delta_n W}\big|^4\Big] + Ck\,\mathbb{E}\Big[\|v^{n+\frac12}\|^4_{\mathbb{L}^2}\Big] + Ck\,\mathbb{E}\Big[\sum_{\ell=-1}^{1} \mathfrak{E}^2(u^{n+\ell}, v^{n+\ell})\Big] \\
&\leq \widehat{\alpha}^4\,C_g^4\,k^3\,\mathbb{E}\Big[\mathfrak{E}^2(u^n, v^n)\Big] + Ck\,\mathbb{E}\Big[\sum_{\ell=-1}^{1} \mathfrak{E}^2(u^{n+\ell}, v^{n+\ell})\Big]\,.
\end{aligned}$$

Now we insert the estimates from parts **a)** and **b)** into (4.20), and sum over $1 \leq n \leq N-1$. Then, for all $k \leq k_0 \equiv k_0(C_{\mathrm{L}}, C_g)$ assertion (4.3) for $p = 1$ follows from the implicit version of the discrete Gronwall lemma.

3) Proof of (4.3) **for** $p \geq 2$**.** Starting from the identity (4.20), we multiply $\delta_1\mathfrak{E}^{2^{p-1}}(u^{n+1}, v^{n+1}) + \delta_2\big[\mathfrak{E}^{2^{p-1}}(u^{n+1}, v^{n+1}) + \mathfrak{E}^{2^{p-1}}(u^{n-1}, v^n)\big]$ on both sides, and then take expectations. We may then follow the same argument as in **2)** to settle the assertion.

4) Proof of (4.4)**.** Let $\widehat{\alpha} = 1$. Suppose $\sigma_2(v) \equiv 0 \equiv F_2(v)$ in **(A3)** and $\beta = 0$. We combine both equations in the scheme (1.9)–(1.10) to get

$$\begin{aligned}
\big[u^{\ell+1} - u^\ell\big] - \big[u^\ell - u^{\ell-1}\big] &= k^2\Delta u^{\ell,1/2} + k\,\sigma(u^\ell)\Delta_\ell W + \widehat{\alpha}k\,D_u\sigma(u^\ell)v^\ell\widehat{\Delta_\ell W} \\
&\quad + \frac{k^2}{2}\big[3F(u^n) - F(u^{n-1})\big]
\end{aligned} \tag{4.22}$$

for all $1 \leq \ell \leq N$. Now sum over the first n steps, and define $\overline{u}^{n+1} := \sum_{\ell=1}^n u^{\ell+1}$ to get

$$\begin{aligned}
\big[u^{n+1} - u^n\big] - k^2\Delta\overline{u}^{n,1/2} &= \big[u^1 - u^0\big] + k\sum_{\ell=1}^{n}\sigma(u^\ell)\Delta_\ell W + \widehat{\alpha}k\sum_{\ell=1}^{n} D_u\sigma(u^\ell)v^\ell\widehat{\Delta_\ell W} \\
&\quad + \frac{k^2}{2}\sum_{\ell=1}^{n}\big[3F(u^\ell) - F(u^{\ell-1})\big]\,.
\end{aligned} \tag{4.23}$$

Multiply both sides with $u^{n+1/2}$ and use integration by parts to get

$$\begin{aligned}&\frac{1}{2}\Big[\|u^{n+1}\|^2_{\mathbb{L}^2} - \|u^n\|^2_{\mathbb{L}^2}\Big] + k^2\big(\nabla \overline{u}^{n,1/2}, \nabla u^{n+1/2}\big)\\ &= \big(u^1 - u^0, u^{n+1/2}\big) + k\Big(\sum_{\ell=1}^{n} \sigma(u^\ell)\Delta_\ell W, u^{n+1/2}\Big) + \widehat{\alpha} k\Big(\sum_{\ell=1}^{n} D_u\sigma(u^\ell)v^\ell \widehat{\Delta_\ell W}, u^{n+1/2}\Big)\\ &\quad + \frac{k^2}{2}\sum_{\ell=1}^{n}\Big(3F(u^\ell) - F(u^{\ell-1}), u^{n+1/2}\Big) =: \mathfrak{K}_1^n + \mathfrak{K}_2^n + \mathfrak{K}_3^n + \mathfrak{K}_4^n\,.\end{aligned} \tag{4.24}$$

We observe that the last term in the left-hand side may be written as

$$k^2\big(\nabla \overline{u}^{n,1/2}, \nabla u^{n+1/2}\big) = \frac{k^2}{4}\Big(\nabla[\overline{u}^{n+1} + \overline{u}^{n-1}], \nabla[\overline{u}^{n+1} - \overline{u}^{n-1}]\Big) = \frac{k^2}{4}\Big[\|\nabla \overline{u}^{n+1}\|^2_{\mathbb{L}^2} - \|\nabla \overline{u}^{n-1}\|^2_{\mathbb{L}^2}\Big].$$

Taking expectation on both sides leads to

$$\frac{1}{2}\mathbb{E}\Big[\|u^{n+1}\|^2_{\mathbb{L}^2} - \|u^n\|^2_{\mathbb{L}^2}\Big] + \frac{k^2}{4}\mathbb{E}\Big[\|\nabla \overline{u}^{n+1}\|^2_{\mathbb{L}^2} - \|\nabla \overline{u}^{n-1}\|^2_{\mathbb{L}^2}\Big] = \sum_{j=1}^{4}\mathbb{E}\big[\mathfrak{K}_j^n\big]\,. \tag{4.25}$$

Since $u^1 - u_0 = kv^1$, by $\mathbf{(B1)}_i$ we infer

$$\mathbb{E}\big[\mathfrak{K}_1^n\big] \leq \frac{1}{k}\mathbb{E}\big[\|u^1 - u_0\|^2_{\mathbb{L}^2}\big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^2_{\mathbb{L}^2}\big] \leq Ck + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^2_{\mathbb{L}^2}\big]\,. \tag{4.26}$$

Using the Itô isometry and **(A3)** we infer

$$\begin{aligned}\mathbb{E}\big[\mathfrak{K}_2^n\big] &\leq k\sum_{\ell=1}^{n}\mathbb{E}\Big[\|\sigma(u^\ell)\|^2_{\mathbb{L}^2}|\Delta_\ell W|^2\Big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^2_{\mathbb{L}^2}\big]\\ &\leq Ck^2C_{\mathtt{L}}^2\sum_{\ell=1}^{n}\mathbb{E}\Big[1 + \|u^\ell\|^2_{\mathbb{L}^2}\Big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^2_{\mathbb{L}^2}\big]\,.\end{aligned} \tag{4.27}$$

Using item **4.** of Remark 1, and **(A4)** for $m = 1$ we infer

$$\begin{aligned}\mathbb{E}\big[\mathfrak{K}_3^n\big] &\leq k\sum_{\ell=1}^{n}\mathbb{E}\Big[\|D_u\sigma(u^\ell)v^\ell\|^2_{\mathbb{L}^2}\big|\widehat{\Delta_\ell W}\big|^2\Big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^2_{\mathbb{L}^2}\big]\\ &\leq k^4C_g^2\sum_{\ell=1}^{n}\mathbb{E}\big[\|v^\ell\|^2_{\mathbb{L}^2}\big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^2_{\mathbb{L}^2}\big]\,.\end{aligned} \tag{4.28}$$

Since $v^\ell = \frac{1}{k}\big[u^\ell - u^{\ell-1}\big]$, we further estimate (4.28) by

$$\leq k^2C_g^2\sum_{\ell=1}^{n}\mathbb{E}\big[\|u^\ell\|^2_{\mathbb{L}^2} + \|u^{\ell-1}\|^2_{\mathbb{L}^2}\big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^2_{\mathbb{L}^2}\big]\,.$$

Using **(A3)** we estimate $\mathbb{E}\big[\mathfrak{K}_4^n\big]$ by

$$\mathbb{E}\big[\mathfrak{K}_4^n\big] \leq Ck^2C_{\mathtt{L}}^2\sum_{\ell=1}^{n}\mathbb{E}\Big[1 + \|u^\ell\|^2_{\mathbb{L}^2} + \|u^{\ell-1}\|^2_{\mathbb{L}^2}\Big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^2_{\mathbb{L}^2}\big]\,. \tag{4.29}$$

We insert these estimates into (4.25) and sum over $1 \leq n \leq N-1$. Then, for all $k \leq k_0 \equiv k_0(C_{\mathtt{L}}, C_g)$ and by the implicit version of the discrete Gronwall lemma, there exists a constant $C > 0$ such that the assertion (4.4) holds.

5) Proof of (4.5) **for** $p = 1$**.** To simplify technicalities, we put $F \equiv 0$. Let us denote $\widetilde{\mathfrak{E}}(u^n, \overline{u}^n) := \Big[\frac{1}{2}\|u^n\|^2_{\mathbb{L}^2} + \frac{k^2}{4}\|\nabla \overline{u}^n\|^2_{\mathbb{L}^2}\Big]$. Then we can rewrite (4.25) as

$$\widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1}) - \widetilde{\mathfrak{E}}(u^n, \overline{u}^{n-1}) = \mathfrak{K}_1^n + \mathfrak{K}_2^n + \mathfrak{K}_3^n \,. \tag{4.30}$$

Multiply both sides with $\widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1})$, using binomial formula and taking expectation we obtain

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\Big[\widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1}) - \widetilde{\mathfrak{E}}^2(u^n, \overline{u}^{n-1})\Big] + \frac{1}{2}\mathbb{E}\Big[\big|\widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1}) - \widetilde{\mathfrak{E}}^2(u^n, \overline{u}^{n-1})\big|^2\Big] \\ &= \mathbb{E}\Big[\mathfrak{K}_1^n\, \widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1})\Big] + \mathbb{E}\Big[\mathfrak{K}_2^n\, \widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1})\Big] + \mathbb{E}\Big[\mathfrak{K}_3^n\, \widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1})\Big] \,. \end{aligned} \tag{4.31}$$

Using Young's inequality, and arguing similarly to (4.26) shows

$$\begin{aligned} \mathbb{E}\Big[\mathfrak{K}_1^n\, \widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1})\Big] &\leq \frac{1}{k^3}\mathbb{E}\Big[\|u^1 - u_0\|^4_{\mathbb{L}^2}\Big] + k^2\mathbb{E}\big[\|u^{n+1/2}\|^4_{\mathbb{L}^2}\big] + k\,\mathbb{E}\Big[\widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1})\Big] \\ &\leq Ck + Ck\,\mathbb{E}\Big[\widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1}) + \widetilde{\mathfrak{E}}^2(u^n, \overline{u}^n)\Big] \,. \end{aligned} \tag{4.32}$$

By adding and subtracting $\widetilde{\mathfrak{E}}(u^n, \overline{u}^{n-1})$, and using **(A3)**, we estimate the second term on the right-hand side of (4.31) by

$$\begin{aligned} &\mathbb{E}\Big[\mathfrak{K}_2^n\,\big(\widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1}) - \widetilde{\mathfrak{E}}(u^n, \overline{u}^{n-1})\big)\Big] + \mathbb{E}\Big[\mathfrak{K}_2^n\, \widetilde{\mathfrak{E}}(u^n, \overline{u}^{n-1})\Big] \\ &\leq \mathbb{E}\big[|\mathfrak{K}_2^n|^2\big] + \frac{1}{4}\,\mathbb{E}\Big[\big|\widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1}) - \widetilde{\mathfrak{E}}(u^n, \overline{u}^{n-1})\big|^2\Big] \\ &\quad + Ck^2C_{\mathrm{L}}^2\sum_{\ell=1}^{n}\,\mathbb{E}\Big[1 + \|u^\ell\|^4_{\mathbb{L}^2}\Big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^4_{\mathbb{L}^2}\big] + Ck\,\mathbb{E}\big[\widetilde{\mathfrak{E}}^2(u^n, \overline{u}^{n-1})\big] \\ &\leq Ck^2C_{\mathrm{L}}^2\sum_{\ell=1}^{n}\,\mathbb{E}\Big[1 + \widetilde{\mathfrak{E}}^2(u^\ell, \overline{u}^\ell)\Big] + Ck\,\mathbb{E}\big[\widetilde{\mathfrak{E}}^2(u^n, \overline{u}^{n-1})\big] \\ &\quad + \frac{1}{4}\,\mathbb{E}\Big[\big|\widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1}) - \widetilde{\mathfrak{E}}(u^n, \overline{u}^{n-1})\big|^2\Big] \,, \end{aligned} \tag{4.33}$$

where the last term in the right-hand side may be absorbed on the left-hand side of (4.31). By item **4.** of Remark 1, and **(A4)** for $m = 1$ we estimate

$$\begin{aligned} &\mathbb{E}\Big[\mathfrak{K}_3^n\, \widetilde{\mathfrak{E}}(u^{n+1}, \overline{u}^{n+1})\Big] \leq \frac{1}{k}\,\mathbb{E}\big[|\mathfrak{K}_3^n|^2\big] + Ck\,\mathbb{E}\Big[\widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1})\Big] \\ &\leq \tilde{C}_g^4\sum_{\ell=1}^{n}\mathbb{E}\Big[\|v^\ell\|^4_{\mathbb{L}^2}\big|\widehat{\Delta_\ell W}\big|^4\Big] + Ck\,\mathbb{E}\big[\|u^{n+1/2}\|^4_{\mathbb{L}^2}\big] + Ck\,\mathbb{E}\Big[\widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1})\Big] \\ &\leq C_g^4k^6\frac{1}{k^4}\sum_{\ell=1}^{n}\mathbb{E}\Big[\|u^\ell\|^4_{\mathbb{L}^2} + \|u^{\ell-1}\|^4_{\mathbb{L}^2}\Big] + Ck\,\mathbb{E}\Big[\widetilde{\mathfrak{E}}^2(u^n, \overline{u}^n) + \widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1})\Big] \\ &\leq Ck^2\sum_{\ell=1}^{n}\mathbb{E}\Big[\widetilde{\mathfrak{E}}^2(u^\ell, \overline{u}^\ell) + \widetilde{\mathfrak{E}}^2(u^{\ell-1}, \overline{u}^{\ell-1})\Big] + Ck\,\mathbb{E}\Big[\widetilde{\mathfrak{E}}^2(u^n, \overline{u}^n) + \widetilde{\mathfrak{E}}^2(u^{n+1}, \overline{u}^{n+1})\Big] \,. \end{aligned} \tag{4.34}$$

Now we insert the estimates into (4.31), and sum over $1 \leq n \leq N - 1$. Then, for all $k \leq k_0 \equiv k_0(C_{\mathrm{L}}, C_g)$ assertion (4.5) for $p = 1$ follows from the implicit version of the discrete Gronwall lemma.

6) Proof of (4.5) **for** $p \geq 2$. Starting from the identity (4.31), we multiply $\widetilde{\mathfrak{E}}^{2^{p-1}}(u^{n+1}, v^{n+1})$ in both sides, and then take the expectation. We may then follow the same argument as in **5)** to settle the assertion.

□

5. Strong Rates of Convergence for $(\widehat{\alpha}, \beta)-$scheme

We prove convergence rate $\mathcal{O}(k^{1/2})$ for the iterates $\{(u^n, v^n)\}_{n\geq 1}$ of the $(\widehat{\alpha}, \beta)-$scheme for $\widehat{\alpha} \in \{0, 1\}$; if additionally $\sigma_2(v) \equiv 0 \equiv F_2(v)$ in **(A3)** holds, we may put $\beta = 0$, and

a) the convergence rate improves to $\mathcal{O}(k)$ for iterates $\{u^n\}_{n\geq 1}$ in case $\widehat{\alpha} = 0$, and
b) to $\mathcal{O}(k^{3/2})$ in case $\widehat{\alpha} = 1$.

For the convergence analysis, we need the following assumption on (u^1, v^1).

(B2) Along with **(A1)**$_{ii}$ and **(B1)**$_{ii}$, let $u^0 = u(0)$ and $v^0 = v(0)$, and (u^1, v^1) satisfy

$$\Big(\mathbb{E}\Big[\|u(t_1) - u^1\|^2_{\mathbb{H}^1} + \|v(t_1) - v^1\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} = \mathcal{O}\big(k^{1/2}\big).$$

Theorem 5.1. *Let (u, v) be the strong solution of (1.3) with $A = -\Delta$. Let $\{(u^n, v^n)\}_{n\geq 1}$ be the iterates from $(\widehat{\alpha}, \beta)-$scheme for $k \leq k_0(C_{\rm L}, C_g)$ sufficiently small, $\widehat{\alpha} \in \{0, 1\}$, and $0 \leq \beta < \frac{1}{2}$. Then, under the hypotheses* **(A1)**$_{iii}$, **(A2)**, **(A3)**, *and* **(A4)**, **(A5)** *for $m = 1, 2$, and* **(B2)**, *there exists $C > 0$ such that*

$$\max_{1\leq n\leq N} \Big(\mathbb{E}\Big[\|u(t_n) - u^n\|^2_{\mathbb{H}^1} + \|v(t_n) - v^n\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} \leq Ck^{1/2}\,. \tag{5.1}$$

For the following, additionally suppose $\sigma_2(v) \equiv 0 \equiv F_2(v)$ in **(A3)** *and that the initial data u^0, u^1, v^0 satisfy*

$$\Big(\mathbb{E}\Big[\|u(t_1) - u^1\|^2_{\mathbb{L}^2}\Big] + \frac{1}{k^2}\,\mathbb{E}\Big[\|kv_0 - (u^1 - u^0)\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} = \mathcal{O}\big(k^{3/2}\big)\,. \tag{5.2}$$

(*i*) *Consider the $(0, 0)-$scheme and assume* **(A1)**$_{iii}$, **(A2)**, **(A3)**, *and* **(A4)**, **(A5)** *for $m = 1, 2$, and* **(B2)**. *Then there exists $C > 0$ such that*

$$\max_{1\leq n\leq N} \Big(\mathbb{E}\Big[\|u(t_n, \cdot) - u^n\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} + \frac{1}{2}\Big(\mathbb{E}\Big[k\sum_{j=1}^{n}\|\nabla\big[u(t_j, \cdot) - u^j\big]\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} \leq Ck\,.$$

(*ii*) *Consider th $(1, 0)-$scheme and assume* **(A1)**$_{iv}$, **(A2)**, **(A3)**, *and* **(A4)**, **(A5)** *for $m = 1, 2, 3$, and* **(B2)**. *Then, there exists $C > 0$ such that*

$$\max_{1\leq n\leq N} \Big(\mathbb{E}\Big[\|u(t_n, \cdot) - u^n\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} + \frac{1}{2}\Big(\mathbb{E}\Big[k\sum_{j=1}^{n}\|\nabla\big[u(t_j, \cdot) - u^j\big]\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} \leq Ck^{3/2}\,.$$

The following remark discusses the realizability of (5.2), and key tools to verify this theorem.

Remark 2. 1. *In Section 6, we choose $(u^0, v^0) = (u(0), v(0))$, together with*

$$u^1 = u_0 + k\,v_0 + k^2\sigma(u_0)\Delta_0 W \qquad \textit{and} \qquad v^1 = v_0 + k\sigma(u_0)\Delta_0 W\,. \tag{5.3}$$

We now prove that (5.2) *holds in this case: first, we consider* (1.3) *in integral form on* $[0, t_1]$,

$$\begin{cases} u(t_1) = u_0 + \int_0^{t_1} v(s)\,\mathrm{d}s \\ v(s) = v_0 + \int_0^s \Delta u(\tau)\mathrm{d}\tau + \int_0^s F(u(\tau))\mathrm{d}\tau + \int_0^s \sigma(u(\tau))\mathrm{d}W(\tau)\,, \quad 0 \le \tau \le s\,, \end{cases} \tag{5.4}$$

and insert (5.4)$_2$ *into (5.4)*$_1$*; a change of order of integration then gives*

$$\begin{aligned} u(t_1) &= u_0 + t_1 v_0 + \int_0^{t_1}\int_0^s \Delta u(\tau)\,\mathrm{d}\tau\mathrm{d}s + \int_0^{t_1}\int_0^s F(u(\tau))\,\mathrm{d}\tau\mathrm{d}s + \int_0^{t_1}\int_0^s \sigma(u(\tau))\mathrm{d}W(\tau)\,\mathrm{d}s \\ &= u_0 + kv_0 + \int_0^{t_1}\int_\tau^{t_1} \mathrm{d}s\,\Delta u(\tau)\,\mathrm{d}\tau + \int_0^{t_1}\int_\tau^{t_1} \mathrm{d}s\,F(u(\tau))\mathrm{d}\tau + \int_0^{t_1}\int_\tau^{t_1} \mathrm{d}s\,\sigma(u(\tau))\mathrm{d}W(\tau)\,. \end{aligned} \tag{5.5}$$

Thus,

$$\begin{aligned} u(t_1) = u_0 + kv_0 &+ \int_0^{t_1} (t_1 - \tau)\Delta u(\tau)\,\mathrm{d}\tau + \int_0^{t_1} (t_1 - \tau)F(u(\tau))\,\mathrm{d}\tau \\ &+ \int_0^{t_1} (t_1 - \tau)\sigma(u(\tau))\mathrm{d}W(\tau)\,. \end{aligned} \tag{5.6}$$

Subtracting $(5.3)_1$ *from* (5.6) *we infer*

$$\begin{aligned} u(t_1) - u^1 &= \int_0^{t_1} (t_1 - \tau)\Delta u(\tau)\,\mathrm{d}\tau + \int_0^{t_1} (t_1 - \tau)F(u(\tau))\,\mathrm{d}\tau \\ &\quad + \int_0^{t_1} (t_1 - \tau)\sigma(u(\tau))\mathrm{d}W(\tau) - k^2\sigma(u_0)\big(W(t_1) - W(0)\big)\,. \end{aligned}$$

By **(A3)**, *Itô isometry, Lemma 3.2* $(i), (ii)$ *and Lemma 3.3* (i) *we infer*

$$\begin{aligned} \mathbb{E}\Big[\|u(t_1) - u^1\|_{\mathbb{L}^2}^2\Big] &\le Ck^2 \int_0^{t_1} \mathbb{E}\Big[\|\Delta u(\tau)\|_{\mathbb{L}^2}^2\Big]\,\mathrm{d}\tau + Ck^2 \int_0^{t_1} \mathbb{E}\Big[\|F(u(\tau))\|_{\mathbb{L}^2}^2\Big]\,\mathrm{d}\tau \\ &\quad + Ck^2 \int_0^{t_1} \mathbb{E}\Big[\|\sigma(u(\tau)) - \sigma(u_0)\|_{\mathbb{L}^2}^2\Big]\,\mathrm{d}s \le Ck^3\,. \end{aligned} \tag{5.7}$$

Similarly, by $(5.3)_1$, $\mathbf{(A1)}_i$ *and Itô isometry we get*

$$\mathbb{E}\Big[\|kv_0 - (u^1 - u^0)\|_{\mathbb{L}^2}^2\Big] \le k^4\,\mathbb{E}\Big[\|\sigma(u_0)\|_{\mathbb{L}^2}^2 |\Delta_0 W|^2\Big] \le Ck^5\,. \tag{5.8}$$

Thus, combining (5.7) *and* (5.8) *we get the assertion* (5.2) *for* u^1*. To validate the choice of* v^1 *in* $(5.3)_2$*, we use* $\mathbf{(A1)}_i$ *and Itô isometry to get*

$$\mathbb{E}\Big[\|kv_0 - (u^1 - u^0)\|_{\mathbb{L}^2}^2\Big] = \mathbb{E}\Big[\|kv_0 - kv^1\|_{\mathbb{L}^2}^2\Big] = k^2\mathbb{E}\Big[\|v^1 - v_0\|_{\mathbb{L}^2}^2\Big] \le k^4\,\mathbb{E}\Big[\|\sigma(u_0)\|_{\mathbb{L}^2}^2|\Delta_0 W|^2\Big] \le Ck^5\,,$$

which settles the assertion (5.2).

2. *For* $\widetilde{\alpha} \neq 0$*, the additional noise term in* (1.10) *improves the accuracy of the* $(\widehat{\alpha}, 0)-$*scheme, where* $\widetilde{\Delta_n W}$ *is approximated by* $\widehat{\Delta_n W}$*. By* (1.11)*,* (1.12)*, and the fact that* $t_{n,\ell+1} - t_{n,\ell} = k^2$*, we estimate the distance between* $\widetilde{\Delta_n W}$ *and* $\widehat{\Delta_n W}$ *as*

$$\begin{aligned} \mathbb{E}\Big[\big|\widetilde{\Delta_n W} - \widehat{\Delta_n W}\big|^2\Big] &= \mathbb{E}\Big[\Big| - \int_{t_n}^{t_{n+1}} W(s)\,\mathrm{d}s + k^2 \sum_{\ell=1}^{k^{-1}} W(t_{n,\ell})\Big|^2\Big] \\ &= \mathbb{E}\Big[\Big|\sum_{\ell=1}^{k^{-1}} \int_{t_{n,\ell}}^{t_{n,\ell+1}} \big(W(s) - W(t_{n,\ell})\big)\,\mathrm{d}s\Big|^2\Big]\,. \end{aligned}$$

By the independence property of the increment $\Delta_n W$, we further estimate

$$\leq k \sum_{\ell=1}^{k^{-1}} \int_{t_{n,\ell}}^{t_{n,\ell+1}} \mathbb{E}\Big[\big|W(s) - W(t_{n,\ell})\big|^2\Big]\,\mathrm{d}s \leq k \sum_{\ell=1}^{k^{-1}} \int_{t_{n,\ell}}^{t_{n,\ell+1}} (s - t_{n,\ell})\,\mathrm{d}s \leq Ck^4\,. \tag{5.9}$$

3. *The basic estimate is* (5.1)*, which will be given in part* **1)** *in the proof below. Its derivation uses the Hölder estimates in Lemma 3.3 for* (u, v) *in* strong *norms. The strategy of proof is similar to the one used in the stability analysis for* $(\widehat{\alpha}, \beta)-$*scheme in Section 4; see item* **1.** *in Remark 1: the central term to estimate is* $T_4^{(n)}$ *in* (5.13)*, in which we replace the increments* $e_v^{n+1} - e_v^n$ *via the* error equation (5.11) *to obtain terms which are scaled by k, or the stochastic increments* $\Delta_n W$ *and* $\widetilde{\Delta_n W}$*. The order limiting term then is* $T_{4,1}^{(n,4)}$ *in* (5.16)*, which may be traced back to the noise term* σ*, which may depend on v as well. In this case (only), the* additional term $-k^{2+\beta}\Delta v^{n+1/2}$ *in Scheme 1 is needed to control the effect of noise: see the additional term on the left-hand side of* (5.12) *to* e.g. *bound the corresponding term in* (5.14)*.*

The verification of assertions (i) and (ii) differs completely from this strategy: it starts with the reformulation (5.17) that leads to the error identity (5.20), which then is tested with $e_u^{n+1/2}$*; the noise part may here be estimated in a straight manner.*

4. *Part* **2)** *in the proof below is conceptually motivated from arguments in* [7]*; however, their realization in the stochastic setting differs considerably. We remark that estimate* (5.1) *is needed in* (5.24) *to verify assertion* (*i*) *— next to Lemma 2.1 to bound the quadrature error of the trapezoidal rule for integrands with limited regularity; see term* $I_4^{\ell,n}$ *in* (5.21)*.*

5. *If* $\widetilde{\alpha} = 0$*, the estimate* (5.23) *for term* $I_3^{\ell,n}$ *in* (5.21) *restricts the order, and assertion* (*i*) *follows; the improvement* (*ii*) *uses* $\widetilde{\alpha} = 1$, s.t. *this term* $I_3^{\ell,n}$ *gives way to the sum* $I_{3;1}^{\ell,n} + I_{3;2}^{\ell,n}$ *in* (5.28)*, which are both of higher order; see* **a)– b)** *in part* **3)** *in the proof below.*

6. *For* $\sigma \equiv \sigma(u, v)$ or $F \equiv F(u, v)$*, neither assertion* (*i*) *nor* (*ii*) *in Theorem 5.1 may be concluded, due to the restricted Hölder regularity properties of v opposed to u.*

In this setting, either σ or F *in* (5.17) *in the proof below would depend on v as well, and thus would modify corresponding terms in* (5.20)*. For* $\sigma \equiv \sigma(u, v)$*, (a modified version of)* **(A3)** *would additionally create a term* $Ck^2 \sum_{\ell=1}^{n} \mathbb{E}\big[\|e_v^\ell\|_{\mathbb{L}^2}^2\big]$ *on the right-hand side of* (5.22)*, which may not be handled via Gronwall's lemma to lift the order. For* $F \equiv F(u, v)$*, the argument in* (5.25) *fails, which rests on Lemma 2.1, and the Hölder continuity of* $v = \partial_t u$*.*

7. *In the proof of* (5.1)*, where* $\sigma \equiv \sigma(u, v)$ *and* $F \equiv F(u, v)$*, we do not require the discrete energy bounds proved in Lemma 4.1. We only require the energy bounds proved in Lemma 3.2. This is possible, since we can add and subtract* $\nabla u(t_n)$ *or* $v(t_n)$ *whenever* L^2*-norm of* ∇u^n *or* v^n *appears. However, the higher moment bounds in energy norm (proved in Lemma 4.1) are required to show the improved convergence order* $\mathcal{O}(k^{3/2})$ *in the proof of* (*ii*) *of Theorem 5.1.*

Proof of Theorem 5.1. **1) Proof of** (5.1). For simplicity, we here give the proof for $F \equiv 0$. Correspondingly, let (u, v) solve (1.3), and $\{(u^n, v^n)\}_{n\in\mathbb{N}}$ solves $(\widehat{\alpha}, \beta)-$scheme, and (u, v) solves (1.3). We denote by $e_u^n := u(t_n) - u^n$ and $e_v^n := v(t_n) - v^n$ error iterates, which are zero on the boundary and solve

$$e_u^{n+1} - e_u^n = k\, e_v^{n+1} + \int_{t_n}^{t_{n+1}} \big(v(s) - v(t_{n+1})\big)\,\mathrm{d}s\,, \tag{5.10}$$

$$\begin{aligned} e_v^{n+1} - e_v^n &= k\,\Delta e_u^{n,1/2} + \int_{t_n}^{t_{n+1}} \Delta\Big[\frac{2u(s) - [u(t_{n+1}) + u(t_{n-1})]}{2}\Big]\,\mathrm{d}s \\ &\quad - k^{2+\beta}\,\Delta e_v^{n+\frac12} + \frac{k^{2+\beta}}{2}\,\Delta\big[v(t_{n+1}) + v(t_n)\big] \\ &\quad + \int_{t_n}^{t_{n+1}} \big[\sigma\big(u(s), v(s)\big) - \sigma\big(u^n, v^{n-\frac12}\big)\big]\,\mathrm{d}W(s) - \widehat{\alpha}\, D_u\sigma(u^n, v^{n-\frac12})\, v^n\, \widehat{\Delta_n W}\,. \end{aligned} \tag{5.11}$$

We multiply (5.11) with $e_v^{n+\frac12}$ and use (5.10) to get

$$\frac12\Big[\|e_v^{n+1}\|^2_{\mathbb{L}^2} - \|e_v^n\|^2_{\mathbb{L}^2}\Big] + \frac14\Big[\|\nabla e_u^{n+1}\|^2_{\mathbb{L}^2} - \|\nabla e_u^{n-1}\|^2_{\mathbb{L}^2}\Big] + k^{2+\beta}\|\nabla e_v^{n+\frac12}\|^2_{\mathbb{L}^2} \le \sum_{j=1}^{5} T_j^{(n)}\,, \tag{5.12}$$

where

$$\begin{aligned} T_1^{(n)} &:= \int_{t_n}^{t_{n+1}} \Big(\nabla\big[v(s) - v(t_{n+1})\big], \nabla e_u^{n,\frac12}\Big)\,\mathrm{d}s + \int_{t_{n-1}}^{t_n} \Big(\nabla\big[v(s) - v(t_n)\big], \nabla e_u^{n,\frac12}\Big)\,\mathrm{d}s\,, \\ T_2^{(n)} &:= -\int_{t_n}^{t_{n+1}} \Big(\nabla\Big[\frac{2u(s) - [u(t_{n+1}) + u(t_{n-1})]}{2}\Big], \nabla e_v^{n+\frac12}\Big)\,\mathrm{d}s\,, \\ T_3^{(n)} &:= \frac{k^{2+\beta}}{2}\Big(\nabla\big[v(t_{n+1}) + v(t_n)\big], \nabla e_v^{n+\frac12}\Big)\,, \\ T_4^{(n)} &:= \Big(\int_{t_n}^{t_{n+1}} \Big[\sigma\big(u(s), v(s)\big) - \sigma(u^n, v^{n-\frac12})\Big]\,\mathrm{d}W(s), e_v^{n+\frac12}\Big)\,, \\ T_5^{(n)} &:= -\,\widehat{\alpha}\,\Big(D_u\sigma(u^n, v^{n-\frac12}) v^n\, \widehat{\Delta_n W}, e_v^{n+\frac12}\Big)\,. \end{aligned}$$

We estimate the expectation of each term on the right-hand side of (5.12). By Lemma 3.3 (*iii*), we infer

$$\mathbb{E}\big[T_1^{(n)} + T_2^{(n)}\big] \le Ck^2 + Ck\,\mathbb{E}\Big[\|\nabla e_u^{n+1}\|^2_{\mathbb{L}^2} + \|\nabla e_u^{n-1}\|^2_{\mathbb{L}^2}\Big] + Ck\,\mathbb{E}\Big[\|e_v^{n+1}\|^2_{\mathbb{L}^2} + \|e_v^n\|^2_{\mathbb{L}^2}\Big]\,.$$

We use Lemma 3.2 (*ii*) to estimate

$$\mathbb{E}\big[T_3^{(n)}\big] \le \frac{k^{2+\beta}}{2}\mathbb{E}\big[\|\nabla e_v^{n+\frac12}\|^2_{\mathbb{L}^2}\big] + Ck^{2+\beta}\,.$$

By properties of $\Delta_n W$ we rewrite the term $T_4^{(n)}$ as

$$\begin{aligned} \mathbb{E}\big[T_4^{(n)}\big] &= \frac12\,\mathbb{E}\Big[\Big(\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, e_v^{n+1} - e_v^n\Big)\Big] \\ &\quad + \frac12\,\mathbb{E}\Big[\Big(\int_{t_n}^{t_{n+1}} \big[\sigma\big(u(s), v(s)\big) - \sigma\big(u(t_n), v(t_{n-1/2})\big)\big]\mathrm{d}W(s), e_v^{n+1} - e_v^n\Big)\Big] \\ &:= T_{4,1}^{(n)} + T_{4,2}^{(n)}\,. \end{aligned} \tag{5.13}$$

In order to estimate $T_{4,1}^{(n)}$, we use equation (5.11) to write

$$
\begin{aligned}
T_{4,1}^{(n)} &= \frac{1}{2}\,\mathbb{E}\Big[\Big(\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, k\Delta e_u^{n,1/2}\Big)\Big] \\
&\quad + \frac{1}{2}\,\mathbb{E}\Bigg[\Bigg(\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, \int_{t_n}^{t_{n+1}} \Delta\Big[\frac{2u - [u(t_{n+1}) + u(t_{n-1})]}{2}\Big]\,\mathrm{d}s\Bigg)\Bigg] \\
&\quad + \frac{1}{2}\,\mathbb{E}\Big[\Big(\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, -k^{2+\beta}\,\Delta v^{n+\frac{1}{2}}\Big)\Big] \\
&\quad + \frac{1}{2}\,\mathbb{E}\Bigg[\Bigg(\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, \int_{t_n}^{t_{n+1}} [\sigma(u,v) - \sigma(u^n, v^{n-\frac{1}{2}})]\,\mathrm{d}W(s)\Bigg)\Bigg] \\
&\quad + \frac{\widehat{\alpha}}{2}\,\mathbb{E}\Bigg[\Bigg(\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, D_u\sigma(u^n, v^{n-\frac{1}{2}})\,v^n\,\widehat{\Delta_n W}\Bigg] \\
&:= T_{4,1}^{(n,1)} + T_{4,1}^{(n,2)} + T_{4,1}^{(n,3)} + T_{4,1}^{(n,4)} + T_{4,1}^{(n,5)}\,.
\end{aligned}
$$

We consider $T_{4,1}^{(n,1)}$ first; to properly address the dependence of σ on v, we first restate it with the help of (5.10) and use the fact that $\sigma\big(u(t_n), v(t_{n-1/2})\big) = \sigma(u^n, v^{n-1/2}) = 0$ on $\partial\mathcal{O}$ to obtain

$$
\begin{aligned}
T_{4,1}^{(n,1)} &= \frac{1}{2}\,\mathbb{E}\Big[\Big(\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, k\Delta[e_u^{n+1} - e_u^{n-1}]\Big)\Big] \\
&= -\frac{1}{2}\,\mathbb{E}\Big[\Big(\nabla\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, 2k^2\nabla e_v^{n+1/2}\Big)\Big] \\
&\quad -\frac{1}{2}\,\mathbb{E}\Big[\Big(\nabla\big[\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\big]\Delta_n W, k\nabla\mathcal{R}_v^{n+1/2}\Big)\Big] =: T_{4,1,\mathtt{A}}^{(n,1)} + T_{4,1,\mathtt{B}}^{(n,1)}\,,
\end{aligned}
$$

where $\mathcal{R}_v^{n+1/2} := \int_{t_n}^{t_{n+1}}\big(v(s) - v(t_{n+1})\big)\,\mathrm{d}s + \int_{t_{n-1}}^{t_n}\big(v(s) - v(t_n)\big)\,\mathrm{d}s$. By chain rule, and **(A4)** for $m = 1$ we obtain

$$
\begin{aligned}
T_{4,1,\mathtt{A}}^{(n,1)} \leq -\mathbb{E}\Big[&C_g\big\{2\|\nabla u(t_n)\|_{\mathbb{L}^2} + 2\|\nabla v(t_{n-1/2})\|_{\mathbb{L}^2} + \|\nabla e_u^n\|_{\mathbb{L}^2} + \|\nabla e_v^{n-1/2}\|_{\mathbb{L}^2}\big\}|\Delta_n W| \\
&k^{1-\frac{\beta}{2}} k^{1+\frac{\beta}{2}}\,\|\nabla e_v^{n+1/2}\|_{\mathbb{L}^2}\Big]\,.
\end{aligned}
$$

We apply Young's inequality, Ito isometry, (4.2) and Lemma 3.2 $(i),(ii)$ to further bound $T_{4,1,\mathtt{A}}^{(n,1)}$ by

$$
\begin{aligned}
T_{4,1,\mathtt{A}}^{(n,1)} &\leq C_g^2\,k^{2-\beta}\,\mathbb{E}\Big[\big\{2\|\nabla u(t_n)\|_{\mathbb{L}^2}^2 + 2\|\nabla v(t_{n-1/2})\|_{\mathbb{L}^2}^2 + \|\nabla e_u^n\|_{\mathbb{L}^2}^2 + \|\nabla e_v^{n-1/2}\|_{\mathbb{L}^2}^2\big\}|\Delta_n W|^2\Big] \\
&\quad + \frac{1}{4}k^{2+\beta}\,\mathbb{E}\Big[\|\nabla e_v^{n+1/2}\|_{\mathbb{L}^2}^2\Big] \\
&\leq Ck^{3-\beta} + Ck^{3-\beta}\mathbb{E}[\|\nabla e_u^n\|_{\mathbb{L}^2}^2] + C_g^2\,k^{3-\beta}\,\mathbb{E}\Big[\|\nabla e_v^{n-1/2}\|_{\mathbb{L}^2}^2\Big] + \frac{1}{4}k^{2+\beta}\,\mathbb{E}\Big[\|\nabla e_v^{n+1/2}\|_{\mathbb{L}^2}^2\Big]\,,
\end{aligned}
\tag{5.14}
$$

where the last two terms on the right-hand side may be absorbed on the left-hand side of (5.12) for $k \leq k_0$ sufficiently small, and $\beta < \frac{1}{2}$. Arguing similarly and by Lemma 3.3 (iii) we infer

$$
T_{4,1,\mathtt{B}}^{(n,1)} \leq Ck^{3-\beta} + C_g^2\,k^{3-\beta}\,\mathbb{E}\big[\|\nabla e_v^{n-1/2}\|_{\mathbb{L}^2}^2\big] + Ck^{2+\beta}\,,
\tag{5.15}
$$

where the second term on right-hand side may be absorbed on the left-hand side of (5.12) for $k \leq k_0$ sufficiently small, and $\beta < \frac{1}{2}$.

We now estimate $T_{4,1}^{(n,2)}$: by properties of $\Delta_n W$, **(A3)** and Lemma 3.3 (iii) we get

$$\begin{aligned}
T_{4,1}^{(n,2)} &\le C_{\mathrm{L}} k\, \mathbb{E}\Big[\big(\|\nabla u(t_n) - \nabla u^n\|_{\mathbb{L}^2}^2 + \|v(t_{n-1/2}) - v^{n-1/2}\|_{\mathbb{L}^2}^2\big)|\Delta_n W|^2\Big] \\
&\quad + C \int_{t_n}^{t_{n+1}} \mathbb{E}\Big[\Big\|\Delta\Big[u(s) - \frac{u(t_{n+1}) + u(t_{n-1})}{2}\Big]\Big\|_{\mathbb{L}^2}^2\Big] \\
&\le Ck\, \mathbb{E}\Big[\|\nabla e_u^n\|_{\mathbb{L}^2}^2 + \|e_v^n\|_{\mathbb{L}^2}^2 + \|e_v^{n-1}\|_{\mathbb{L}^2}^2\Big] + Ck^3 .
\end{aligned}$$

Using similar arguments as for the estimate of (5.14) we infer

$$\begin{aligned}
T_{4,1}^{(n,3)} &\le C_g^2 k^{3+\beta} \mathbb{E}\Big[\|\nabla u(t_n)\|_{\mathbb{L}^2}^2 + 2\|\nabla v(t_{n-1/2})\|_{\mathbb{L}^2}^2 + \|\nabla u^n\|_{\mathbb{L}^2}^2 + \|\nabla e_v^{n-1/2}\|_{\mathbb{L}^2}^2\Big] \\
&\quad + \frac{1}{4} k^{2+\beta}\, \mathbb{E}\Big[\|\nabla e_v^{n+1/2}\|_{\mathbb{L}^2}^2 + \|\nabla v(t_{n+1/2})\|_{\mathbb{L}^2}^2\Big] \\
&\le C_g^2\, k^{3+\beta}\, \mathbb{E}\Big[\|\nabla e_v^{n-1/2}\|_{\mathbb{L}^2}^2\Big] + \frac{1}{4} k^{2+\beta}\, \mathbb{E}\Big[\|\nabla e_v^{n+1/2}\|_{\mathbb{L}^2}^2\Big] + Ck^{2+\beta} .
\end{aligned}$$

Using **(A3)**, Lemma 3.3 (ii), and properties of $\Delta_n W$, we estimate

$$\begin{aligned}
T_{4,1}^{(n,4)} &\le C\, \mathbb{E}\Big[\|\sigma(u(t_n), v(t_{n-1/2})) - \sigma(u^n, v^{n-1/2})\|_{\mathbb{L}^2}^2\, |\Delta_n W|^2\Big] \\
&\quad + C \int_{t_n}^{t_{n+1}} \mathbb{E}\Big[\|\sigma\big(u(s), v(s)\big) - \sigma\big(u(t_n), v(t_{n-1/2})\big)\|_{\mathbb{L}^2}^2\Big]\, \mathrm{ds} \\
&\le Ck\, \mathbb{E}\Big[\|\nabla e_u^n\|_{\mathbb{L}^2}^2 + \|e_v^n\|_{\mathbb{L}^2}^2 + \|e_v^{n-1}\|_{\mathbb{L}^2}^2\Big] \qquad (5.16) \\
&\quad + C_{\mathrm{L}} \int_{t_n}^{t_{n+1}} \mathbb{E}\Big[\|\nabla[u - u(t_n)]\|_{\mathbb{L}^2}^2 + \|v - v(t_{n-1/2})\|_{\mathbb{L}^2}^2\Big] \\
&\le Ck\, \mathbb{E}\Big[\|\nabla e_u^n\|_{\mathbb{L}^2}^2 + \|e_v^n\|_{\mathbb{L}^2}^2 + \|e_v^{n-1}\|_{\mathbb{L}^2}^2\Big] + Ck^2 .
\end{aligned}$$

Using **(A3)**, **(A4)** for $m = 1$, item **4.** of Remark 1, and using Lemma 3.2 (i) (due to addition and subtraction of $v(t_n)$ term to v^n) we estimate

$$\begin{aligned}
T_{4,1}^{(n,5)} &\le k\, \mathbb{E}\Big[\|\sigma\big(u(t_n), v(t_{n-1/2})\big) - \sigma(u^n, v^{n-1/2})\|_{\mathbb{L}^2}^2\Big] + \frac{\widehat{\alpha}^2}{4} k^3\, \mathbb{E}\Big[\|D_u \sigma(u^n, v^{n-\frac{1}{2}})\, v^n\|_{\mathbb{L}^2}^2\Big] \\
&\le C_{\mathrm{L}} k\, \mathbb{E}\Big[\|\nabla e_u^n\|_{\mathbb{L}^2}^2 + \|e_v^{n-1/2}\|_{\mathbb{L}^2}^2\Big] + \frac{\widehat{\alpha}^2}{4} C_g^2\, k^3\, \mathbb{E}\big[\|e_v^n\|_{\mathbb{L}^2}^2\big] + Ck^3 .
\end{aligned}$$

Similar arguments, in combination with the Hölder estimates in Section 3.1 may be used to estimate $T_{4,2}^{(n)}$ in (5.13). Now, we estimate the last term in the right-hand side of (5.12). Using **(A4)** for $m = 1$, Itô isometry, and Lemma 3.2 (i) (due to addition and subtraction of $v(t_n)$ term to v^n), we obtain

$$\begin{aligned}
\mathbb{E}\big[T_5^{(n)}\big] &\le \frac{\widehat{\alpha}^2}{k} \mathbb{E}\Big[\|D_u \sigma(u^n, v^{n-\frac{1}{2}}) v^n\|_{\mathbb{L}^2}^2 \big|\widehat{\Delta_n W}\big|^2\Big] + Ck\, \mathbb{E}\big[\|e_v^{n+1}\|_{\mathbb{L}^2}^2 + \|e_v^n\|_{\mathbb{L}^2}^2\big] \\
&\le \widehat{\alpha}^2\, C_g^2\, k^2\, \mathbb{E}\big[\|v^n\|_{\mathbb{L}^2}^2\big] + Ck\, \mathbb{E}\big[\|e_v^{n+1}\|_{\mathbb{L}^2}^2 + \|e_v^n\|_{\mathbb{L}^2}^2\big] \le Ck^2 + Ck\, \mathbb{E}\big[\|e_v^{n+1}\|_{\mathbb{L}^2}^2 + \|e_v^n\|_{\mathbb{L}^2}^2\big] .
\end{aligned}$$

We now insert these estimates into (5.12), for which we apply expectations, and sum over iteration steps. The implicit version of the discrete Gronwall lemma then yields the assertion, again provided $k \le k_0$ is sufficiently small.

2) Proof of (i)**.** Suppose $\sigma_2(v) \equiv 0 \equiv F_2(v)$ in **(A3)**, and $\widehat{\alpha} = 0$. We combine both equations in the $(0,0)-$scheme,

$$\big[u^{\ell+1} - u^{\ell}\big] - \big[u^{\ell} - u^{\ell-1}\big] = k^2 \Delta u^{\ell,1/2} + \frac{k^2}{2}\big[3F(u^n) - F(u^{n-1})\big] + k\sigma(u^\ell)\Delta_\ell W \tag{5.17}$$

for all $1 \leq \ell \leq N$. Now sum over the first n steps, and define $\overline{u}^{n+1} := \sum_{\ell=1}^{n} u^{\ell+1}$. We arrive at

$$\big[u^{n+1} - u^n\big] - k^2 \Delta \overline{u}^{n,1/2} = \big[u^1 - u^0\big] + \frac{k^2}{2}\sum_{\ell=1}^{n}\big[3F(u^\ell) - F(u^{\ell-1})\big] + k\sum_{\ell=1}^{n}\sigma(u^\ell)\Delta_\ell W\,. \tag{5.18}$$

We proceed correspondingly with (3.2), which we integrate in time: thanks to (3.1), we get $(0 \leq \lambda \leq \mu \leq T)$

$$\begin{aligned}\big[u(\mu) - u(\lambda)\big] &- \int_\lambda^\mu \int_0^s \Delta u(\xi)\,\mathrm{d}\xi\mathrm{d}s \\ &= [\mu - \lambda]v_0 + \int_\lambda^\mu \int_0^s F\big(u(\xi)\big)\,\mathrm{d}\xi\mathrm{d}s + \int_\lambda^\mu \int_0^s \sigma\big(u(\xi)\big)\,\mathrm{d}W(\xi)\mathrm{d}s\,.\end{aligned} \tag{5.19}$$

For every $s \in [t_n, t_{n+1}]$, we write $\int_0^s \cdot\,\mathrm{d}s = \sum_{\ell=0}^{n_s-1}\int_{t_\ell}^{t_{\ell+1}} \cdot\,\mathrm{d}s$, where $n_s = \lfloor \frac{s}{k} \rfloor$. Setting $\mu = t_{n+1}$, $\lambda = t_n$ in (5.19), subtracting (5.18) from (5.19) then leads to

$$\begin{aligned}\big[e_u^{n+1} - e_u^n\big] - k^2\Delta\overline{e}_u^{n,1/2} &= \big[kv_0 - (u^1 - u^0)\big] + k\sum_{\ell=1}^{n}\big[\sigma\big(u(t_\ell)\big) - \sigma(u^\ell)\big]\Delta_\ell W \\ &\quad + \int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\big[\sigma\big(u(\xi)\big) - \sigma\big(u(t_\ell)\big)\big]\,\mathrm{d}W(\xi)\mathrm{d}s \\ &\quad + \int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\Delta\Big[u(\xi) - \frac{u(t_{\ell+1}) + u(t_{\ell-1})}{2}\Big]\,\mathrm{d}\xi\mathrm{d}s \\ &\quad + \int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}} F\big(u(\xi)\big) - \frac{1}{2}\big[3F\big(u(t_\ell)\big) - F\big(u(t_{\ell-1})\big)\big]\,\mathrm{d}\xi\mathrm{d}s \\ &\quad + \frac{k^2}{2}\sum_{\ell=1}^{n}\Big(3\big[F\big(u(t_\ell)\big) - F(u^\ell)\big] - \big[F\big(u(t_{\ell-1})\big) - F(u^{\ell-1})\big]\Big)\,,\end{aligned} \tag{5.20}$$

which holds for all $n \geq 1$. Now multiply with $e_u^{n+1/2}$, and observe that

$$\begin{aligned}k^2\Big(\nabla\overline{e}_u^{n,1/2}, \nabla e_u^{n+1/2}\Big) &= \frac{k^2}{4}\Big(\nabla\overline{e}_u^{n+1} + \nabla\overline{e}_u^{n-1}, \nabla[\overline{e}_u^{n+1} - \overline{e}_u^{n-1}]\Big) \\ &= \frac{k^2}{4}\Big[\|\nabla\overline{e}_u^{n+1}\|_{\mathbb{L}^2}^2 - \|\nabla\overline{e}_u^{n-1}\|_{\mathbb{L}^2}^2\Big]\,.\end{aligned}$$

Using this in (5.20) then leads to

$$
\begin{aligned}
&\frac{1}{2}\Big[\|e_u^{n+1}\|_{\mathbb{L}^2}^2 - \|e_u^{n}\|_{\mathbb{L}^2}^2\Big] + \frac{k^2}{4}\Big[\|\nabla\overline{e}_u^{n+1}\|_{\mathbb{L}^2}^2 - \|\nabla\overline{e}_u^{n-1}\|_{\mathbb{L}^2}^2\Big] \\
&\quad= \big(kv_0 - [u^1-u^0], e_u^{n+1/2}\big) + k\Big(\sum_{\ell=1}^{n}\big[\sigma\big(u(t_\ell)\big) - \sigma(u^\ell)\big]\Delta_\ell W, e_u^{n+1/2}\Big) \\
&\qquad+\Big(\int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\big[\sigma\big(u(\xi)\big) - \sigma\big(u(t_\ell)\big)\big]\mathrm{d}W(\xi)\mathrm{d}s, e_u^{n+1/2}\Big) \\
&\qquad-\int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\Big(\nabla\Big[u(\xi) - \frac{u(t_{\ell+1})+u(t_{\ell-1})}{2}\Big], \nabla e_u^{n+1/2}\Big)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad+\int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\Big(F(u) - \frac{1}{2}\big[3F\big(u(t_\ell)\big) - F\big(u(t_{\ell-1})\big)\big], e_u^{n+1/2}\Big)\,\mathrm{d}\xi\mathrm{d}s \\
&\qquad+\frac{k^2}{2}\sum_{\ell=1}^{n}\Big(3\big[F\big(u(t_\ell)\big) - F(u^\ell)\big] - \big[F(u(t_{\ell-1})) - F(u^{\ell-1})\big], e_u^{n+1/2}\Big) \\
&\quad=: I_1^n + I_2^{\ell,n} + \ldots + I_6^{\ell,n}\,.
\end{aligned}
\tag{5.21}
$$

We estimate the six terms in (5.21) separately. From (5.2) we infer

$$
\mathbb{E}\big[I_1^n\big] \le \frac{C}{k}\,\mathbb{E}\Big[\|kv_0 - [u^1-u^0]\|_{\mathbb{L}^2}^2\Big] + Ck\,\mathbb{E}\Big[\|e_u^{n+1/2}\|_{\mathbb{L}^2}^2\Big] \le Ck^4 + Ck\,\mathbb{E}\Big[\|e_u^{n+1/2}\|_{\mathbb{L}^2}^2\Big]\,.
$$

For $I_2^{\ell,n}$, by Itô isometry, and **(A3)**, we have

$$
\begin{aligned}
\mathbb{E}\big[I_2^{\ell,n}\big] &\le Ck\,\mathbb{E}\big[\|e_u^{n+1/2}\|_{\mathbb{L}^2}^2\big] + k\,\mathbb{E}\Big[\Big\|\sum_{\ell=1}^{n}\big[\sigma\big(u(t_\ell)\big) - \sigma(u^\ell)\big]\Delta_\ell W\Big\|_{\mathbb{L}^2}^2\Big] \\
&\le Ck\,\mathbb{E}\big[\|e_u^{n+1/2}\|_{\mathbb{L}^2}^2\big] + \tilde{C}_{\mathrm{L}}k^2\sum_{\ell=1}^{n}\mathbb{E}\big[\|e_u^{\ell}\|_{\mathbb{L}^2}^2\big]\,.
\end{aligned}
\tag{5.22}
$$

The term $I_3^{\ell,n}$ can be controlled by Itô isometry, **(A3)**, and Lemma 3.3 (i) as

$$
\begin{aligned}
\mathbb{E}\big[I_3^{\ell,n}\big] &= \mathbb{E}\Big[\Big(\int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\big[\sigma\big(u(\xi)\big) - \sigma\big(u(t_\ell)\big)\big]\,\mathrm{d}W(\xi)\mathrm{d}s, e_u^{n+1/2}\Big)\Big] \\
&\le \tilde{C}_{\mathrm{L}}\int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\mathbb{E}\Big[\big\|u(\xi) - u(t_\ell)\big\|_{\mathbb{L}^2}^2\Big]\mathrm{d}\xi + Ck\,\mathbb{E}\big[\|e_u^{n+1/2}\|_{\mathbb{L}^2}^2\big] \\
&\le \tilde{C}_{\mathrm{L}}k^3 + Ck\,\mathbb{E}\big[\|e_u^{n+1}\|_{\mathbb{L}^2}^2 + \|e_u^{n}\|_{\mathbb{L}^2}^2\big]\,.
\end{aligned}
\tag{5.23}
$$

We use Lemma 2.1 to estimate $\mathbb{E}[I_4^{\ell,n}]$. For this, we choose $f(\xi) = \mathbb{E}\big[(\nabla u(\xi), \nabla e_u^{n+1/2})\big]$ for all $\xi \in [t_n, t_{n+1}]$. By Lemma 3.3 (iii), we have $\gamma = \frac{1}{2}$ in (2.1),

$$
\Big|\mathbb{E}\Big[\Big(\nabla[v(t) - v(s)], \nabla e_u^{n+1/2}\Big)\Big]\Big| \le C\Big(\mathbb{E}\big[\|\nabla e_u^{n+1/2}\|_{\mathbb{L}^2}^2\big]\Big)^{1/2}|t-s|^{\frac{1}{2}}\,.
$$

As a consequence,

$$\int_{t_\ell}^{t_{\ell+1}} \mathbb{E}\Big[\Big(\nabla\Big[u(\xi) - \frac{u(t_{\ell+1})+u(t_{\ell-1})}{2}\Big], \nabla e_u^{n+1/2}\Big)\Big]\,\mathrm{d}\xi \le C\Big(\mathbb{E}\big[\big\|\nabla e_u^{n+1/2}\big\|_{\mathbb{L}^2}^2\big]\Big)^{\frac{1}{2}} k^{\frac{5}{2}}\,.$$

This estimate then yields

$$\mathbb{E}\big[I_4^{\ell,n}\big] \le Ck^{\frac{5}{2}}\Big(\mathbb{E}\big[\big\|\nabla e_u^{n+1/2}\big\|_{\mathbb{L}^2}^2\big]\Big)^{1/2} \le \frac{k^2}{4}\mathbb{E}\big[\|\nabla e_u^{n+1/2}\|_{\mathbb{L}^2}^2\big] + Ck^3, \tag{5.24}$$

by using Young's inequality. By (5.1) the leading term on the right-hand side is again bounded above by Ck^3.

Next, we turn to $\mathbb{E}[I_5^{\ell,n}]$: as a first step, we split it into two parts,

$$\begin{aligned}\mathbb{E}[I_5^{\ell,n}] &= \int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}} \mathbb{E}\Big[\Big(F\big(u(\xi)\big) - \frac{1}{2}\big[F\big(u(t_\ell)\big) + F\big(u(t_{\ell+1})\big)\big], e_u^{n+1/2}\Big)\Big]\,\mathrm{d}\xi\mathrm{d}s \\ &+ \frac{k^2}{2}\sum_{\ell=1}^{n}\mathbb{E}\Big[\Big(F\big(u(t_{\ell+1})\big) - 2F\big(u(t_\ell)\big) + F\big(u(t_{\ell-1})\big), e_u^{n+1/2}\Big)\Big] =: \mathbb{E}\big[I_{5;1}^{\ell,n} + I_{5;2}^{\ell,n}\big]\,.\end{aligned}$$

To handle these two terms, we use Lemma 2.1 with $f(\xi) = \mathbb{E}\big[\big(F\big(u(\xi)\big), e_u^{n+1/2}\big)\big]$ where $\xi \in [t_n, t_{n+1}]$, and verify $\gamma = \frac{1}{2}$ in (2.1): by **(A4)** for $m = 1, 2$, the chain-rule and the mean-value theorem

$$\begin{aligned}&\Big|\big(D_tF\big(u(t)\big) - D_tF\big(u(s)\big), e_u^{n+1/2}\big)\Big| \\ &= \Big|\Big(D_uF\big(u(t)\big)v(t) - D_uF\big(u(s)\big)v(s), e_u^{n+1/2}\Big)\Big| \\ &= \Big|\Big(\big(D_uF(u(t)) - D_uF(u(s))\big)v(t) + D_uF(u(s))(v(t)-v(s)), e_u^{n+1/2}\Big)\Big| \\ &= \Big|\Big(\big(\underline{D_u^2F}(u(t)-u(s))\big)(v(t)) + D_uF(u(s))(v(t)-v(s)), e_u^{n+1/2}\Big)\Big| \\ &\le \tilde{C}_g\,\|u(t)-u(s)\|_{\mathbb{H}^1}\|v(t)\|_{\mathbb{H}^1}\|e_u^{n+1/2}\|_{\mathbb{L}^2} + \tilde{C}_g\,\|v(t)-v(s)\|_{\mathbb{L}^2}\|e_u^{n+1/2}\|_{\mathbb{L}^2}\,.\end{aligned} \tag{5.25}$$

where $\underline{D_u^2F} := D_u^2F(\widetilde{u}_\rho)$ and $\widetilde{u}_\rho := \rho u(t) + (1-\rho)u(s)$, for some $\rho \in [0,1]$. Lemma 3.3 (*ii*) then establishes $\gamma = \frac{1}{2}$ in (2.1), and so Lemma 2.1 yields

$$\mathbb{E}\big[I_{5;1}^{\ell,n}\big] \le Ck^{\frac{5}{2}}C\Big(\mathbb{E}\big[\big\|e_u^{n+1/2}\big\|_{\mathbb{L}^2}^2\big]\Big)^{\frac{1}{2}} \le Ck^4 + k\,\mathbb{E}\big[\big\|e_u^{n+1/2}\big\|_{\mathbb{L}^2}^2\big]\,.$$

In order to estimate $\mathbb{E}[I_{5;2}^{\ell,n}]$, we may write for some $\theta \in (0,1)$

$$\begin{aligned}F(u(t_{\ell+1})) &= F(u(t_\ell)) + D_uF(u(t_\ell))(u(t_{\ell+1}) - u(t_\ell)) \\ &+ \frac{1}{2}\Big(D_u^2F\big(u(t_\ell) + \theta(u(t_{\ell+1}) - u(t_\ell))\big)(u(t_{\ell+1}) - u(t_\ell))\Big)\big(u(t_{\ell+1}) - u(t_\ell)\big)\,,\end{aligned}$$

and

$$\begin{aligned}F(u(t_{\ell-1})) &= F(u(t_\ell)) + D_uF(u(t_\ell))(u(t_{\ell-1}) - u(t_\ell)) \\ &+ \frac{1}{2}\Big(D_u^2F\big(u(t_\ell) + \theta(u(t_{\ell-1}) - u(t_\ell))\big)(u(t_{\ell-1}) - u(t_\ell))\Big)\big(u(t_{\ell-1}) - u(t_\ell)\big)\,.\end{aligned}$$

Then, adding the above two terms we get

$$
\begin{aligned}
&F\big(u(t_{\ell+1})\big) - 2F\big(u(t_\ell)\big) + F\big(u(t_{\ell-1})\big) \\
&= \underline{D_u F}\big(u(t_{\ell+1}) - 2u(t_\ell) + u(t_{\ell-1})\big) + \frac{1}{2}\big(\underline{D_u^2 F}\,(u(t_{\ell+1}) - u(t_\ell))\big)(u(t_{\ell+1}) - u(t_\ell)) \\
&\quad + \frac{1}{2}\big(\overline{D_u^2 F}\,(u(t_{\ell-1}) - u(t_\ell))\big)(u(t_{\ell-1}) - u(t_\ell))\,,
\end{aligned}
$$

where $\underline{D_u F} := D_u F(u(t_\ell))$, $\underline{D_u^2 F} := D_u^2 F\big(u(t_\ell)+\theta(u(t_{\ell+1})-u(t_\ell))\big)$ and $\overline{D_u^2 F} := D_u^2 F\big(u(t_\ell)+\theta(u(t_{\ell-1}) - u(t_\ell))\big)$. We begin with the first term on the right-hand side: first, by the mean value theorem, there exist $\zeta_1, \zeta_2 \in [0,1]$, such that

$$
u(t_{\ell+1}) - u(t_\ell) = kv\big(\zeta_1 t_\ell + [1-\zeta_1]t_{\ell+1}\big)\,, \qquad -\big[u(t_\ell) - u(t_{\ell-1})\big] = -kv\big(\zeta_2 t_{\ell-1} + [1-\zeta_2]t_\ell\big)\,.
$$

Hence, Lemma 3.3 (ii) settles $\mathcal{O}(k^{\frac{3}{2}})$ for this term. If combined with Lemma 3.3 (i), **(A4)** for $m = 1, 2$, we can conclude

$$
\mathbb{E}[I_{5;2}^{\ell,n}] \leq Ck^4 + k\,\mathbb{E}\big[\|e_u^{n+1/2}\|_{\mathbb{L}^2}^2\big]\,.
$$

Finally, by **(A3)** we infer

$$
\mathbb{E}[I_6^{\ell,n}] \leq Ck^2\Big(\sum_{\ell=1}^{n}\mathbb{E}\Big[\|e_u^\ell\|_{\mathbb{L}^2}^2 + \|e_u^{\ell-1}\|_{\mathbb{L}^2}^2\Big] + \mathbb{E}\big[\|e_u^{n+1/2}\|_{\mathbb{L}^2}^2\big]\Big)\,.
$$

Now we combine all the above estimates in (5.21) in summarized form, then the implicit version of the discrete Gronwall lemma yields assertion (i).

3) Proof of (ii)**.** Similar to (5.18), we have for $\widehat{\alpha} = 1$

$$
\begin{aligned}
&\big[u^{n+1} - u^n\big] - k^2\Delta\overline{u}^{n,1/2} - \big[u^1 - u^0\big] \\
&= k\sum_{\ell=1}^{n}\sigma(u^\ell)\Delta_\ell W + \widehat{\alpha}k\sum_{\ell=1}^{n}D_u\sigma(u^\ell)v^\ell\widehat{\Delta_\ell W} + \frac{k^2}{2}\sum_{\ell=1}^{n}\big[3F(u^\ell) - F(u^{\ell-1})\big]\,.
\end{aligned}
\tag{5.26}
$$

So the additional term on the right-hand side of the error equation (5.20) is

$$
\widehat{\alpha}k\sum_{\ell=1}^{n} -D_u\sigma(u^\ell)v^\ell\widetilde{\Delta_\ell W} + \widehat{\alpha}k\sum_{\ell=1}^{n} -D_u\sigma(u^\ell)v^\ell\big[(\widehat{\Delta_\ell W} - \widetilde{\Delta_\ell W})\big] := \mathfrak{I}_{3,\mathtt{A}}^{\ell,n} + \mathfrak{I}_{3,\mathtt{B}}^{\ell,n}\,.
\tag{5.27}
$$

We now follow the argumentation in **2)**: multiplication with $e_u^{n+1/2}$ of the modified error equation (5.20) then leads to (5.21), where $\mathfrak{I}_{3,\mathtt{A}}^{\ell,n}$ is merged with $I_3^{\ell,n}$ which may be written by the sum of two terms

$$
\begin{aligned}
I_{3,\mathtt{A}_1}^{\ell,n} + I_{3,\mathtt{A}_2}^{\ell,n} =: &\Big(\int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\Big[D_u\sigma\big(u(t_\ell)\big)v(t_\ell) - D_u\sigma(u^\ell)v^\ell\Big](\xi - t_\ell)\,\mathrm{d}W(\xi)\mathrm{d}s, e_u^{n+1/2}\Big) \\
&+ \Big(\int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\Big[\sigma\big(u(\xi)\big) - \sigma\big(u(t_\ell)\big) - D_u\sigma\big(u(t_\ell)\big)v(t_\ell)(\xi - t_\ell)\Big]\,\mathrm{d}W(\xi)\mathrm{d}s, e_u^{n+1/2}\Big)\,.
\end{aligned}
\tag{5.28}
$$

We independently bound the other error terms in (5.21) in this modified setting:

a) To bound $\mathbb{E}[I^{\ell,n}_{3;\mathtt{A}_1}]$ in (5.28), we use Itô isometry, the mean-value theorem, **(A4)** for $m = 1, 2$, to get

$$
\begin{aligned}
\mathbb{E}\big[I^{\ell,n}_{3;\mathtt{A}_1}\big] &\le k\mathbb{E}\Big[\frac{1}{k}\cdot\Big\|\int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\int_{t_\ell}^{t_{\ell+1}}\big[D_u\sigma\big(u(t_\ell)\big)v(t_\ell) - D_u\sigma(u^\ell)v^\ell\big] \\
&\qquad\qquad \times(\xi - t_\ell)\,\mathrm{d}W(\xi)\,\mathrm{d}s\Big\|^2_{\mathbb{L}^2}\Big] + k\,\mathbb{E}\big[\|e_u^{n+1/2}\|^2_{\mathbb{L}^2}\big] \\
&\le \int_{t_n}^{t_{n+1}}\sum_{\ell=1}^{n}\mathbb{E}\Big[\int_{t_\ell}^{t_{\ell+1}}\big\|D_u\sigma\big(u(t_\ell)\big)v(t_\ell) - D_u\sigma(u^\ell)v^\ell\big\|^2_{\mathbb{L}^2} \\
&\qquad\qquad \times(\xi - t_\ell)^2\,d\xi\,\mathrm{d}s\Big] + k\,\mathbb{E}\big[\|e_u^{n+1/2}\|^2_{\mathbb{L}^2}\big] \\
&\le Ck^4\sum_{\ell=1}^{n}\mathbb{E}\Big[\big\|D_u\sigma\big(u(t_\ell)\big)v(t_\ell) - D_u\sigma(u^\ell)v^\ell\big\|^2_{\mathbb{L}^2}\Big] + k\,\mathbb{E}\big[\|e_u^{n+1/2}\|^2_{\mathbb{L}^2}\big] \\
&\le Ck^4\sum_{\ell=1}^{n}\Big(\mathbb{E}\Big[\|e_v^\ell\|^2_{\mathbb{L}^2}\Big] + \mathbb{E}\Big[\widetilde{I^{\ell,n}_{3;\mathtt{A}_1}}\Big]\Big) + k\,\mathbb{E}\Big[\|e_u^{n+1/2}\|^2_{\mathbb{L}^2}\Big],
\end{aligned}
\tag{5.29}
$$

where $\widetilde{I^{\ell,n}_{3;\mathtt{A}_1}} := \big\|[D_u\sigma\big(u(t_\ell)\big) - D_u\sigma(u^\ell)]v\big(t_\ell\big)\big\|^2_{\mathbb{L}^2}$. In order to handle the first term in the right-hand side, we estimate (5.29) further by

$$
\le Ck^2\sum_{\ell=1}^{n}\Big(\mathbb{E}\Big[\|e_u^\ell\|^2_{\mathbb{L}^2} + \|e_u^{\ell-1}\|^2_{\mathbb{L}^2}\Big] + k^2\mathbb{E}\Big[\widetilde{I^{\ell,n}_{3;\mathtt{A}_1}}\Big]\Big) + k\,\mathbb{E}\Big[\|e_u^{n+1/2}\|^2_{\mathbb{L}^2}\Big].
$$

We estimate the second term in the right-hand side as

$$
\begin{aligned}
Ck^4\sum_{\ell=1}^{n}\mathbb{E}\Big[\widetilde{I^{\ell,n}_{3;\mathtt{A}_1}}\Big] &\le Ck^4\sum_{\ell=1}^{n}\mathbb{E}\Big[\|e_u^\ell\|^2_{\mathbb{L}^2}\|v(t_\ell)\|^2_{\mathbb{L}^\infty}\Big] \\
&\le Ck\cdot k^3\sum_{\ell=1}^{n}\mathbb{E}\Big[\|e_u^\ell\|_{\mathbb{L}^2}\|e_u^\ell\|_{\mathbb{L}^2}\|v(t_\ell)\|^2_{\mathbb{L}^\infty}\Big] \\
&\le Ck^2\sum_{\ell=1}^{n}\mathbb{E}\Big[\|e_u^\ell\|^2_{\mathbb{L}^2}\Big] + Ck^6\sum_{\ell=1}^{n}\mathbb{E}\Big[\|e_u^\ell\|^4_{\mathbb{L}^2}\Big] + Ck^6\sum_{\ell=1}^{n}\mathbb{E}\Big[\|v(t_\ell)\|^8_{\mathbb{L}^\infty}\Big],
\end{aligned}
\tag{5.30}
$$

where the last term on the right-hand side is bounded by Ck^5 due to Lemma 3.2 (iii). The second term on the right-hand side is bounded further by $Ck^6\sum_{\ell=1}^{n}\mathbb{E}\Big[\|u(t_\ell)\|^4_{\mathbb{L}^2} + \|u^\ell\|^4_{\mathbb{L}^2}\Big]$, which may be bounded by Ck^5, thanks to Lemma 3.2 (i) for $p = 2$, and (4.5).

b) Now consider $\mathbb{E}[I^{\ell,n}_{3;\mathtt{A}_2}]$. Let $\xi \in [t_n, t_{n+1}]$; we use the mean-value theorem twice, **(A4)** for $m = 1, 2$, to conclude

$$
\begin{aligned}
&\big\|\sigma\big(u(\xi)\big) - \sigma\big(u(t_\ell)\big) - D_u\sigma\big(u(t_\ell)\big)v(t_\ell)(\xi - t_\ell)\big\|^2_{\mathbb{L}^2} \\
&= \Big\|[D_u\sigma(\widetilde{u}_\zeta) - D_u\sigma\big(u(t_\ell)\big)]\int_{t_\ell}^{\xi}v(\eta)\,\mathrm{d}\eta + D_u\sigma\big(u(t_\ell)\big)\int_{t_\ell}^{\xi}[v(\eta) - v(t_\ell)]\,\mathrm{d}\eta\Big\|^2_{\mathbb{L}^2} \\
&\le C\|\nabla u(\xi) - \nabla u(t_\ell)\|^4_{\mathbb{L}^2} + Ck^2\sup_{t_\ell\le\xi\le t_{\ell+1}}\|v(\xi) - v(t_\ell)\|^2_{\mathbb{L}^2},
\end{aligned}
\tag{5.31}
$$

where $\widetilde{u}_\zeta = \zeta u(\xi) + (1-\zeta)u(t_\ell)$, for some $\zeta \in [0,1]$. Thus, we have

$$\begin{aligned}\mathbb{E}[I^{\ell,n}_{3;\mathtt{A}_2}] &\leq Ck\,\mathbb{E}\Big[\sup_{t_\ell\leq\xi\leq t_{\ell+1}} \big(\|\nabla u(\xi)-\nabla u(t_\ell)\|^4_{\mathbb{L}^2} + k^2\,\|v(\xi)-v(t_\ell)\|^2_{\mathbb{L}^2}\big)\Big] + k\,\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big] \\ &\leq Ck^4 + k\,\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big]\,.\end{aligned}$$

c) To estimate the term involving $\mathfrak{I}^{\ell,n}_{3,\mathtt{B}}$, which is defined in (5.27), we use Young's inequality to write

$$\mathbb{E}\big[\big(\mathfrak{I}^{\ell,n}_{3,\mathtt{B}}, e^{n+1/2}_u\big)\big] \leq \frac{\widehat{\alpha}^2}{k}k^2\,\mathbb{E}\Big[\Big\|\sum_{\ell=1}^{n} D_u\sigma(u^\ell)v^\ell\big[\widehat{\Delta_\ell W} - \widetilde{\widehat{\Delta_\ell W}}\big]\Big\|^2_{\mathbb{L}^2}\Big] + k\,\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big]\,.$$

Then, we use **(A4)** for $m=1$, and independence of increments $\Delta_n W$ to get

$$\leq \widehat{\alpha}^2\,C_g^2\,k\sum_{\ell=1}^{n}\mathbb{E}\Big[\|D_u\sigma(u^\ell)v^\ell\|^2_{\mathbb{L}^2}\big|\widehat{\Delta_\ell W} - \widetilde{\widehat{\Delta_\ell W}}\big|^2\Big] + k\,\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big]\,.$$

Finally, we use (5.9) and (4.2) of Lemma 4.1 to obtain

$$\leq \widehat{\alpha}^2\,C_g^2\,k^5\sum_{\ell=1}^{n}\mathbb{E}\big[\|v^\ell\|^2_{\mathbb{L}^2}\big] + k\,\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big] \leq Ck^4 + k\,\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big]\,. \tag{5.32}$$

d) We may modify the argument in part **2)** to improve the bound $\mathbb{E}[I^{\ell,n}_4]$ in (5.24). Using integration by parts and using Lemma 3.3 (*iv*) instead, we verify (2.1) of Lemma 2.1 for $\gamma = 1/2$ (by choosing $f(\xi) = \mathbb{E}\big[(\nabla u(\xi), \nabla e^{n+1/2}_u)\big]$ for all $\xi\in[t_n,t_{n+1}]$) to get

$$\Big|\mathbb{E}\Big[\big(\nabla[v(t)-v(s)], \nabla e^{n+1/2}_u\big)\Big]\Big| \leq C\Big(\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big]\Big)^{1/2}|t-s|^{\frac{1}{2}}\,.$$

Using this estimate we infer for $I^{\ell,n}_4$ in (5.21) that

$$\mathbb{E}\big[I^{\ell,n}_4\big] \leq Ck^{\frac{5}{2}}\Big(\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big]\Big)^{1/2} \leq Ck\,\mathbb{E}\big[\|e^{n+1/2}_u\|^2_{\mathbb{L}^2}\big] + Ck^4\,.$$

Thanks to the above estimates in **a)**–**c)**, and after summation over all iteration steps in (5.21) we may then conclude assertion (*ii*). ☐

6. Computational experiments

In this section, we provide computational studies to check

- how essential the assumptions **(A1)**–**(A5)** and **(B1)**–**(B2)** (*i.e.,* needed in Sections 3–5) are in actual computations. In this respect, we computationally study the impact of rough initial data (u_0, v_0) on the discrete dynamics, as well as of drift nonlinearities F (see Example 4).
- If the diffusion $\sigma\equiv\sigma(v)$ and the drift $F\equiv 0$, then there is a reduction of convergence order as proved in (5.1) of Theorem 5.1; see Example 3.
- The diffusion $\sigma\equiv\sigma(u,v) = 0$ on the boundary, and satisfies **(A3)**. Example 5 discusses the effect that noise has, which is non-homogeneous on the boundary, or violates **(A3)**.

- By Theorem 4.1, β in the $(\widehat{\alpha},\beta)$-scheme needs be chosen from $(0,1/2)$ to ensure stable, accurate simulation of (1.3) with $\sigma \equiv \sigma(u,v)$ and $F \equiv F(u,v)$. The simulations in Example 6 evidence a small choice for β for faster Monte Carlo approximation.

We use the lowest order conforming finite element method to simulate the $(\widehat{\alpha},\beta)-$scheme on a regular triangulation $\mathcal{T}_h$ of $\mathcal{O}$; see [2]. Let the finite element space be

$$\mathbb{V}_h := \left\{u_h \in \mathbb{H}_0^1 : \ u_h\big|_K \in \mathcal{P}_1(K) \quad \forall\, K \in \mathcal{T}_h\right\},$$

where $\mathcal{P}_1(K)$ denotes the space of polynomials of degree one on $K \in \mathcal{T}_h$.

As initial data, we choose u^1 and v^1 as

$$u^1 = u_0 + kv_0 + k^2\sigma(u_0)W(t_1), \qquad \text{and} \qquad v^1 = v_0 + k\sigma(u_0)W(t_1)\,, \tag{6.1}$$

where u_0, v_0 (not finite element valued) satisfy assumptions $\mathbf{(A1)}_{iv}$ and **(B2)**. Recall the definitions for $\widetilde{u}^{n,1/2}$ and $\widehat{\Delta_n W}$ in (4.1) and (1.11), respectively. We implement the following scheme:

Scheme 3. *Let* $\widehat{\alpha} \in \{0,1\}$, *and* $0 \leq \beta < \frac{1}{2}$. *Let* $\{t_n\}_{n=0}^N$ *be a mesh of size* $k > 0$ *covering* $[0,T]$, *and* (6.1). *For every* $n \geq 1$, *find a* $[\mathbb{V}_h]^2$*-valued,* $\mathcal{F}_{t_{n+1}}$*-measurable random variable* (u_h^{n+1}, v_h^{n+1}) *such that*

$$\big(u_h^{n+1} - u_h^n, \phi_h\big) = k(v_h^{n+1}, \phi_h) \qquad \forall \phi_h \in \mathbb{V}_h\,, \tag{6.2}$$

$$\begin{aligned}(v_h^{n+1} - v_h^n, \psi_h) = &-k\big(\nabla \widetilde{u}_h^{n,1/2}, \nabla\psi_h\big) + \Big(\sigma(u_h^n, v_h^{n-\frac{1}{2}})\Delta_n W, \psi_h\Big) \\ &+ \widehat{\alpha}\,\Big(D_u\sigma(u_h^n, v_h^{n-\frac{1}{2}})v_h^n\,\widehat{\Delta_n W}, \psi_h\Big) \\ &+ \frac{k}{2}\Big(3F(u_h^n, v_h^n) - F(u_h^{n-1}, v_h^{n-1}), \psi_h\Big) \qquad \forall \psi_h \in \mathbb{V}_h\,.\end{aligned} \tag{6.3}$$

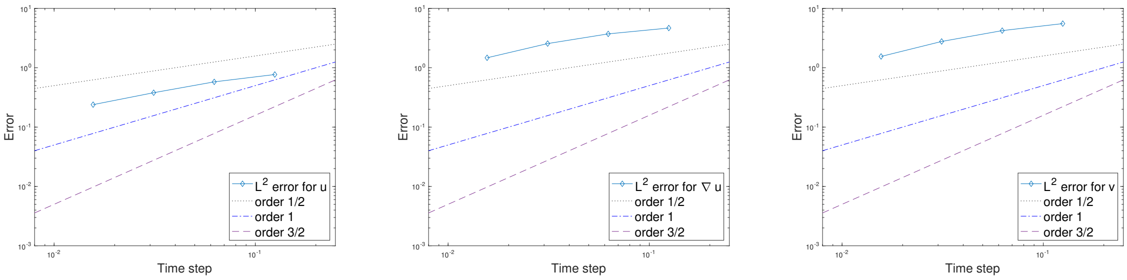
6.1. **Convergence rates.** The numerical experiments are performed using MATLAB. In this section, for all the examples we choose $\mathcal{O} = (0,1)$, $T = 1$, $A = -\Delta$ in (1.3). We choose $u_0(x) = \sin(2\pi x)$ and $v_0(x) = \sin(3\pi x)$, and u^1, v^1 are chosen as in (6.1). A reference solution is computed with a step size $k_{\mathtt{ref}} = 2^{-7}$ and $h_{\mathtt{ref}} = 2^{-7}$ to approximate the exact solution and the sample Wiener processes W. The expected values are approximated by computing averages over $\mathtt{MC} = 3000$ number of samples. The plots are shown for the time steps $k = \{2^{-3}, \cdots, 2^{-6}\}$.

Example 2 in Section 1 provides computational evidence for the improved convergence rate $\mathcal{O}(k^{3/2})$ for the scheme (1.9)–(1.10) with $\widehat{\alpha} = 1$ in the situations where $\sigma \equiv \sigma(u)$. In the following example, we consider $\sigma \equiv \sigma(v)$, and find a convergence rates of $\mathcal{O}(k^{1/2})$ in simulations (A)–(C) of Fig. 6.1, which validates (5.1) of Theorem 5.1. So we observe a reduction of convergence order if compared to Example 2, where $\sigma \equiv \sigma(u)$.

Example 3. *Consider* $\sigma(v) = \frac{3}{2}v$ *and* $F \equiv 0$. *Fig. 6.1 displays convergence studies for the* $(\widehat{\alpha},\beta)-$*scheme for* $\widehat{\alpha} = 1$ *and* $\beta = 1/4$: *the plots* (A)–(C) *of* $\mathbb{L}^2$*-errors in* $u, \nabla u$ *and* v, *respectively, confirm convergence order* $\mathcal{O}(k^{1/2})$; *see* (5.1) *of Theorem 5.1.*

In the following example, we discuss four different cases where

(*i*) $F \equiv F(u,v)$ is non-zero on the boundary, but Lipschitz and $\sigma \equiv \sigma(u)$;



(A) $\mathbb{L}^2$-error for u (B) $\mathbb{L}^2$-error for ∇u (C) $\mathbb{L}^2$-error for v

FIGURE 6.1. **(Example 3)** Rates of convergence of the $\left(1, \frac{1}{4}\right)-$scheme with $\sigma(v) = \frac{3}{2}v$ and $F \equiv 0$.

(ii) $F \equiv F(u, v)$ only Hölder continuous, and $\sigma \equiv \sigma(u)$;
(iii) $F \equiv F(u, v)$ is same as (i), and $\sigma \equiv \sigma(u, v)$ satisfying **(A3)**;
(iv) $F \equiv F(u, v)$ is same as (ii), and $\sigma \equiv \sigma(u, v)$ satisfying **(A3)**.

We observe that although $F \equiv F(u, v)$ violates **(A3)** in (ii), we still get improved convergence rates, but if $\sigma \equiv \sigma(u, v)$, we get the convergence order $\mathcal{O}(k^{1/2})$ as shown in (5.1) of Theorem 5.1.

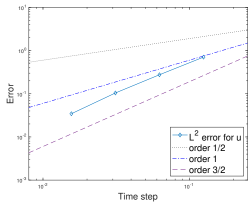
Example 4. *We consider the following cases:*
(i) $\sigma(u) = u$ *and* $F(u, v) = \cos(u) + 2v$;
(ii) $\sigma(u) = u$ *and* $F(u, v) = \sqrt{u} + \sqrt{v + 2}$;
(iii) $\sigma(u, v) = \frac{u}{1+u^2} + v$ *and* $F(u, v) = \cos(u) + 2v$;
(iv) $\sigma(u, v) = \frac{u}{1+u^2} + v$ *and* $F(u, v) = \sqrt{u} + \sqrt{v + 2}$;

The errors are computed via the $(\widehat{\alpha}, \beta)-$*scheme with* $\widehat{\alpha} = 1$ *for* $\beta = 1/4$: *the plots* (A)–(C) *for the problem* (i) *evidence the convergence order* $\mathcal{O}(k^{3/2})$ *for* $u, \nabla u$, *and* $\mathcal{O}(k)$ *for* v. *We observe the same convergence rates for the problem* (ii) *despite the lack of Lipschitzness of* F *which violates* **(A3)**; *see plots* (D)–(F) *of Fig. 6.2. The plots* (G)–(I) *of* $\mathbb{L}^2$*-errors in* $u, \nabla u$ *and* v, *respectively, for the problem* (iii) *and evidence the convergence order* $\mathcal{O}(k^{1/2})$ *as shown in* (5.1) *of Theorem 5.1. We observe the same order of convergence for the problem* (iv); *see plots* (J)–(L) *of Fig. 6.2. Thus, this example shows that the estimate* (5.1) *is sharp in the case of diffusion* $\sigma \equiv \sigma(u, v)$.

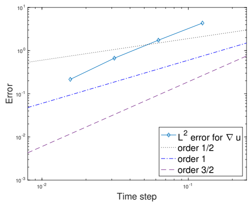
In the next example, we drop the assumption on $\sigma \equiv \sigma(u)$ to be Lipschitz and zero on the boundary to see which of these violations spot the reduction of the convergence order of scheme (1.9)-(1.10).

Example 5. *Let* $F \equiv 0$. *Consider the following cases:*
(i) $\sigma(u) = \frac{1}{1+u^2}$;
(ii) $\sigma(u) = \sqrt{|u|}$.

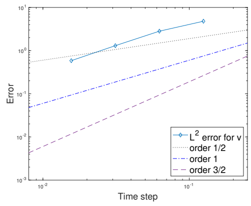
In Fig. 6.3, the errors are computed via the scheme (1.9)–(1.10) *with* $\widehat{\alpha} = 1$. *For problem* (i) *(nonzero boundary), the plots* (A)–(B) *for* $\mathbb{L}^2$*-errors in* $u, \nabla u$, *respectively, show the convergence order* $\mathcal{O}(k^{3/2})$ *and the plot* (C) *for* $\mathbb{L}^2$*-error in* v *shows* $\mathcal{O}(k)$. *For the problem* (ii)



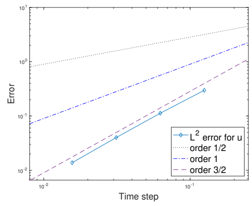
(A) $\mathbb{L}^2$-error for u in (i)



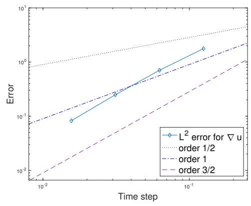
(B) $\mathbb{L}^2$-error for ∇u in (i)



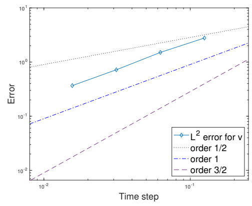
(C) $\mathbb{L}^2$-error for v in (i)



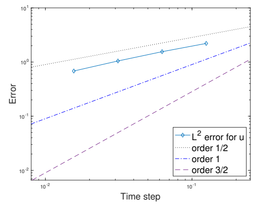
(D) $\mathbb{L}^2$-error for u in (ii)



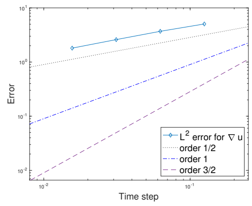
(E) $\mathbb{L}^2$-error for ∇u in (ii)



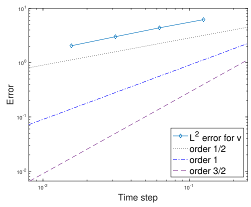
(F) $\mathbb{L}^2$-error for v in (ii)



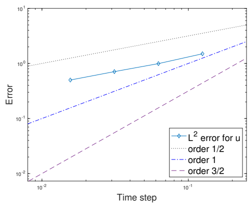
(G) $\mathbb{L}^2$-error for u in (iii)



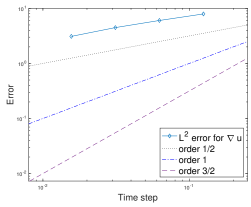
(H) $\mathbb{L}^2$-error for ∇u in (iii)



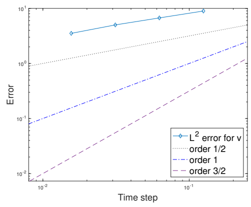
(I) $\mathbb{L}^2$-error for v in (iii)



(J) $\mathbb{L}^2$-error for u in (iv)



(K) $\mathbb{L}^2$-error for ∇u in (iv)



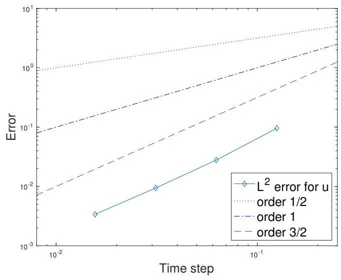
(L) $\mathbb{L}^2$-error for v in (iv)

FIGURE 6.2. **(Example 4)** Rates of convergence of the $\left(1, \frac{1}{4}\right)-$scheme.

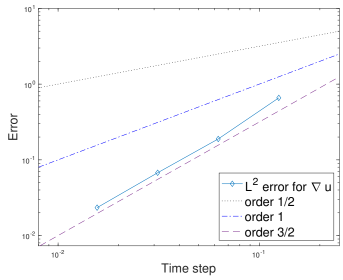
(non-Lipschitz), the convergence rates for $\mathbb{L}^2$*-errors in* $u, \nabla u$ *are reduced to* $\mathcal{O}(k)$*; see plots* (D)–(E)*, but* $\mathbb{L}^2$*-error in* v *remains same as* $\mathcal{O}(k)$*; see plot* (F)*.*

6.2. **Choice of β and required number of MC.**

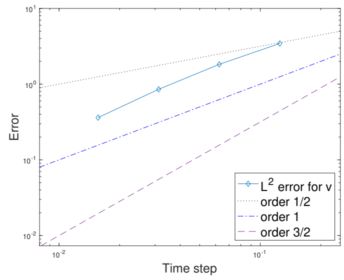
Example 6. *Let* $\mathcal{O} = (0, 1)$*,* $T = 0.5$*,* $A = -\Delta$*,* $F \equiv 0$*,* $\sigma(v) = 5v$*. We compute* W *on the mesh of size* $k = 2^{-12}$ *covering* $[0, 0.5]$*. In the* $(\widehat{\alpha}, \beta)$*-scheme, the term* $\widetilde{u}^{n,\frac{1}{2}} = u^{n,\frac{1}{2}} + \beta k^{1+\beta} v^{n+\frac{1}{2}}$ *involves* β*, where the last term creates an additional numerical dissipation term in* (1.3) *to control discretization effect of the noise. For* $\beta = 0$ *with* $\sigma \equiv \sigma(u)$ *and* $F \equiv F(u)$*, the scheme* (1.9)–(1.10) *is stable, but for general case we require* $\beta \in (0, 1/2)$ *for the stability of the* $(\widehat{\alpha}, \beta)$*-scheme; see Lemma 4.1. For increased value of* β*, stabilization effect vanishes for small* k*. Thus, a smaller choice of* β *is preferred to have the stability of the scheme. The snapshot* (A) *in Fig. 6.4 shows for* $\beta = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$*, that at least* MC $= 400, 600, 800, 1000, 1400,$



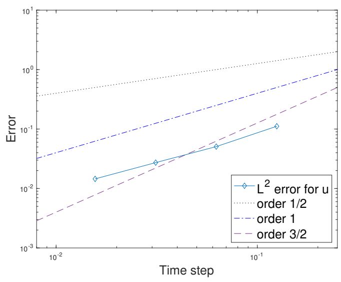
(A) $\mathbb{L}^2$-error for u in (i)



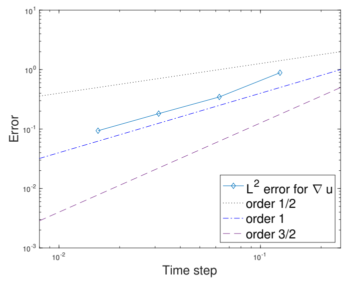
(B) $\mathbb{L}^2$-error for ∇u in (i)



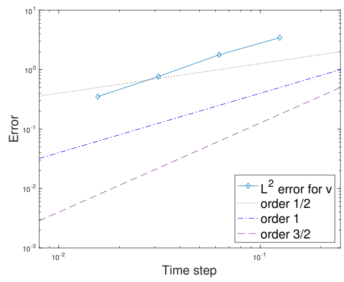
(C) $\mathbb{L}^2$-error for v in (i)



(D) $\mathbb{L}^2$-error for u in (ii)



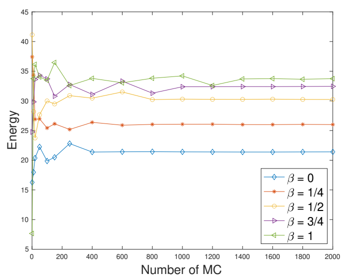
(E) $\mathbb{L}^2$-error for ∇u in (ii)



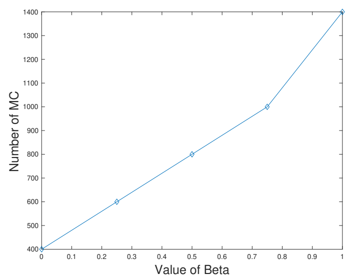
(F) $\mathbb{L}^2$-error for v in (ii)

FIGURE 6.3. **(Example 5)** Rates of convergence of the the scheme (1.9)–(1.10) for $\widehat{\alpha} = 1$.

are needed to have a steady of the energy $\mathcal{E}$ at time $T = 0.5$. The snapshot (B) evidence a higher number of MC *as we increase β to have a steady energy curve.*



(A) Energy for different β



(B) β vs number of MC

FIGURE 6.4. **(Example 6)** $(\widehat{\alpha}, \beta)$-scheme with $\sigma(v) = 5v$, and $F \equiv 0$.

Appendix A. Proof of Lemma 3.2

We exploit the linearity of the drift operator to decompose the solution u of (1.3) with $A = -\Delta$ in the form $u = u_1 + u_2$, where u_1 solves the following PDE

$$\begin{cases} \mathrm{d}\dot{u}_1 - \Delta u_1 \,\mathrm{d}t = F(0,0)\,\mathrm{d}t & \text{in } (0,T)\times\mathcal{O}\,, \\ u_1(0,\cdot) = 0\,, \quad \partial_t u_1(0,\cdot) = 0 & \text{in } \mathcal{O}\,, \\ u_1(t,\cdot) = 0 & \text{on } \partial\mathcal{O},\ \forall\, t\in(0,T)\,, \end{cases} \tag{A.1}$$

where " $\cdot$ " denotes the time derivative, while u_2 solves the SPDE

$$\begin{cases} \mathrm{d}\dot{u}_2 - \Delta u_2 \,\mathrm{d}t = \widehat{F}(u,v)\,\mathrm{d}t + \sigma(u,v)\,\mathrm{d}W(t) & \text{in } (0,T)\times\mathcal{O}\,, \\ u_2(0,\cdot) = u_0\,, \quad \partial_t u_2(0,\cdot) = v_0 & \text{in } \mathcal{O}\,, \\ u_2(t,\cdot) = 0 & \text{on } \partial\mathcal{O},\ \forall\, t\in(0,T)\,, \end{cases} \tag{A.2}$$

where $\widehat{F}(u,v) := F(u,v) - F(0,0)$, and $v = \partial_t u := \partial_t u_1 + \partial_t u_2$. The reason for introducing $\widehat{F}$ is to make the drift term has zero trace in $(\mathrm{A.2})_1$. To prove the regularity results, we use the framework of [8] for (A.1) and we use the Galerkin-based proof for (A.2); see, *e.g.* [3, Ch. 6]. We need some extra assumptions on F and σ (*e.g.* **(A3)–(A5)**) and use different arguments than [3] as we require improved regularity results.

For the argumentation below to work, for the improved regularity, we assume that σ is zero on the boundary, but not F. If this is not assumed, then the subproblem (A.1) will have an extra term $\Big(\int_0^t \sigma(0,0)\,\mathrm{d}W(s), \phi\Big)$ in the right-hand side, and in (A.2), $\sigma(u,v)$ will be replaced by $\widehat{\sigma}(u,v) := \sigma(u,v) - \sigma(0,0)$, which is zero on the boundary. In the next step to prove the higher regularity of the modified (A.1), we need to consider the following transformation, $y(t) = u_1(t) - \int_0^t\int_0^s \sigma(0,0)\,\mathrm{d}W(r)\,\mathrm{d}s$. Now, y solves a randomized PDE with $y = h$ on the boundary, where $h(t) := \int_0^t\int_0^s \sigma(0,0)\,\mathrm{d}W(r)\,\mathrm{d}s$. Since h is of class $\mathrm{C}^{1,\frac12}$ with respect to the time variable, the standard PDE techniques to show the improved regularity may not be applied. This motivates us to assume that σ is zero on the boundary.

Proof of Lemma 3.2. We first prove the improved regularity results for u_1 and use a bootstrapping argument to prove the improved regularity results for u_2.

a) Improved regularity of u_1. By [8, Sec. 7.2], there exists a unique solution $u_1 \in C([0,T];\mathbb{H}^1_0)$ and $\partial_t u_1 \in C([0,T];\mathbb{L}^2)$ to (A.1). By [8, Sec. 7.2], for $m = 1,2,3$, under the assumption **(A5)**, we get $\big(u_1, \partial_t u_1\big) \in L^\infty(0,T;\mathbb{H}^{m+1}) \times L^\infty(0,T;\mathbb{H}^m)$, and we have the following estimate

$$\sup_{0\le t\le T} \Big(\|u_1(t)\|^q_{\mathbb{H}^{m+1}} + \|\partial_t u_1(t)\|^q_{\mathbb{H}^m} \Big) \le C_q\, \|F(0,0)\|^q_{L^2(0,T;\mathbb{H}^m)} \qquad (q\ge 2). \tag{A.3}$$

We will use this result to prove the improved regularity for u_2.

b) Improved regularity of u_2. By [3, Thm. 8.4], there exists a unique $\{\mathcal{F}_t\}_{t\ge0}$−adapted process $(u_2,\partial_t u_2) \in L^2\big(\Omega; C([0,T];\mathbb{H}^1_0)\big) \times L^2\big(\Omega; C([0,T];\mathbb{L}^2)\big)$, which satisfies (A.2) $\mathbb{P}$−a.s.. The proof uses a Galerkin approximation, with $\{\rho_i\}_{i=1}^\infty$ the orthonormal basis of $\mathbb{L}^2$, composed of eigenfunctions of $-\Delta$. For any $n\in\mathbb{N}$, we define the finite dimensional space $\mathbb{H}_n := \mathrm{Span}\{\rho_1,\cdots,\rho_n\}$, and $\mathcal{P}_n$ be the projection from $\mathbb{L}^2$ onto $\mathbb{H}_n$. We define $\Delta_n := \mathcal{P}_n\Delta : \mathbb{H}_n \to \mathbb{H}_n$ and use the mappings $\widehat{F}_n(u_n,v_n) := \mathcal{P}_n\widehat{F}(u_n,v_n) \in \mathbb{H}_n$ and $\sigma_n(u_n,v_n) := \mathcal{P}_n\sigma(u_n,v_n) \in \mathbb{H}_n$ for $(u_n,v_n)\in[\mathbb{H}_n]^2$, such that $u_n = u_{1n} + u_{2n}$, where $u_{1n} := \mathcal{P}_n u_1$, $v_n := v_{1n} + v_{2n} :=$

$\partial_t u_{1n} + \partial_t u_{2n}$ with $u_{2n}(0) = \mathcal{P}_n u_0$ and $v_{2n}(0) = \mathcal{P}_n v_0$, where u_{2n} and v_{2n} satisfy the following approximated system

$$\begin{cases} \mathrm{d}u_{2n} &= v_{2n}\,\mathrm{d}t \\ \mathrm{d}v_{2n} &= \Big(\Delta_n u_{2n} + \widehat{F}_n\big(u_{1n}+u_{2n}, v_{1n}+v_{2n}\big)\Big)\,\mathrm{d}t + \sigma_n\big(u_{1n}+u_{2n}, v_{1n}+v_{2n}\big)\mathrm{d}W(t)\,. \end{cases} \tag{A.4}$$

By [11], there exists a unique $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process (u_{2n}, v_{2n}) on $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$ such that for each $n \in \mathbb{N}$, $(u_{2n}, v_{2n}) \in L^2\big(\Omega; C([0,T]; [\mathbb{H}_n]^2)\big)$ for $(\mathcal{P}_n u_0, \mathcal{P}_n v_0) \in [\mathbb{H}_n]^2$.

1) Bounds: Let $\ell \in \Big\{0, \frac{1}{2}, 1, \frac{3}{2}\Big\}$, which correspond to the parts $(i) - (iv)$ of Lemma 3.2, respectively. Define the map $\Phi_\ell : \mathbb{H}_n \times \mathbb{H}_n \to \mathbb{R}$, where

$$\Phi_\ell(u,v) := \frac{1}{2}\Big[\|\Delta_n^{\ell+\frac{1}{2}} u\|_{\mathbb{L}^2}^2 + \|\Delta_n^\ell v\|_{\mathbb{L}^2}^2\Big]\,.$$

Thus, $D_u\Phi_\ell(u,v), D_v\Phi_\ell(u,v) \in \mathcal{L}(\mathbb{H}_n, \mathbb{R})$. For any $\phi \in \mathbb{H}_n$, we have

$$D_u\Phi_\ell(u,v)(\phi) = \big(\Delta_n^{\ell+\frac{1}{2}} u, \Delta_n^{\ell+\frac{1}{2}}\phi\big) \quad\text{and}\quad D_v\Phi_\ell(u,v)(\phi) = \big(\Delta_n^\ell v, \Delta_n^\ell \phi\big)\,.$$

Applying Itô's formula to the process Φ_ℓ we obtain

$$\begin{aligned} \Phi_\ell\big(u_{2n}(t), v_{2n}(t)\big) &= \Phi_\ell\big(u_{2n}(0), v_{2n}(0)\big) + \int_0^t \Big(\Delta_n^{\ell+\frac{1}{2}} u_{2n}(s), \Delta_n^{\ell+\frac{1}{2}} v_{2n}(s)\Big)\,\mathrm{d}s \\ &+ \int_0^t \Big(\Delta_n^\ell v_{2n}(s), \Delta_n^{\ell+1} u_{2n}(s) + \Delta_n^\ell \widehat{F}_n\big(u_n(s), v_n(s)\big)\Big)\,\mathrm{d}s \\ &+ \int_0^t \Big(\Delta_n^\ell v_{2n}(s), \Delta_n^\ell \sigma_n\big(u_n(s), v_n(s)\big)\,\mathrm{d}W(s)\Big) \\ &+ \frac{1}{2}\int_0^t \big\|\Delta_n^\ell \sigma_n\big(u_n(s), v_n(s)\big)\big\|_{\mathbb{L}^2}^2\,\mathrm{d}s\,, \end{aligned} \tag{A.5}$$

where $u_n = u_{1n} + u_{2n}$ and $v_n = v_{1n} + v_{2n}$. We use different arguments for the cases $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}$, which represnt the parts $(i) - (iv)$ of Lemma 3.2, respectively.

b1) $F \equiv F(u,v)$ **and** $\sigma \equiv \sigma(u,v)$ **for** $\ell = 0$**.** Since $\mathcal{P}_n\sigma(u_n, v_n) = \sum_{i=1}^n \big(\sigma\big(u_n, v_n\big), \rho_i\big)\rho_i$, using **(A3)**, a standard argument gives

$$\begin{aligned} \big\|\sigma_n\big(u_n, v_n\big)\big\|_{\mathbb{L}^2}^2 &\leq \big\|\sigma\big(u_n, v_n\big)\big\|_{\mathbb{L}^2}^2 \leq C_{\mathtt{L}}\Big\{1 + \|\nabla u_n\|_{\mathbb{L}^2}^2 + \|v_n\|_{\mathbb{L}^2}^2\Big\} \\ &\leq C_{\mathtt{L}}\Big\{1 + \|\nabla u_n\|_{\mathbb{L}^2}^2 + \|v_n\|_{\mathbb{L}^2}^2\Big\} \leq C\Big\{1 + \|\Delta_n^{1/2} u_n\|_{\mathbb{L}^2}^2 + \|v_n\|_{\mathbb{L}^2}^2\Big\}\,. \end{aligned}$$

A similar estimate will hold for $\|\widehat{F}_n(u_n, v_n)\|_{\mathbb{L}^2}^2$.

b2) $F \equiv F(u,v)$ **and** $\sigma \equiv \sigma(u,v)$ **for** $\ell = 1/2$**.** Proceeding similarly as before for $\ell = 0$, and using **(A4)** we infer

$$\begin{aligned} &\big\|\Delta_n^{1/2}\sigma_n(u_n, v_n)\big\|_{\mathbb{L}^2}^2 = \sum_{j=1}^n \lambda_j \Big|\Big(\sigma\big(u_n, v_n\big), \rho_j\Big)\Big|^2 \leq \big\|\nabla\sigma\big(u_n, v_n\big)\big\|_{\mathbb{L}^2}^2 \\ &\leq C\Big\{1 + \big\|\partial_u\sigma\big(u_n, v_n\big)(\nabla u_n) + \partial_v\sigma\big(u_n, v_n\big)(\nabla v_n)\big\|_{\mathbb{L}^2}^2\Big\} \leq C\Big\{1 + \|\Delta_n u_n\|_{\mathbb{L}^2}^2 + \|\Delta_n^{1/2} v_n\|_{\mathbb{L}^2}^2\Big\}\,. \end{aligned}$$

A similar estimate will hold for $\big\|\Delta_n^{1/2}\widehat{F}_n\big(u_n, v_n\big)\big\|_{\mathbb{L}^2}^2$. The other terms in the right-hand side of (A.5) can be dealt similarly by the use of Cauchy-Schwarz inequality.

Using the above estimates in **b1)** and **b2)** (for $\ell = 0$ and $\frac{1}{2}$, respectively) in (A.5) we obtain

$$\begin{aligned}\Phi_\ell\big(u_{2n}(t), v_{2n}(t)\big) &\leq \Phi_\ell\big(u_{2n}(0), v_{2n}(0)\big) + C\int_0^t \Big[1 + \|\Delta_n^\ell v_n(s)\|_{\mathbb{L}^2}^2 + \|\Delta_n^{\ell+\frac{1}{2}} u_n(s)\|_{\mathbb{L}^2}^2\Big]\mathrm{d}s \\ &\quad + \int_0^t \Big(\Delta_n^\ell v_{2n}(s), \Delta_n^\ell \sigma_n\big(u_n(s), v_n(s)\big)\mathrm{d}W(s)\Big). \end{aligned} \tag{A.6}$$

Using the definition of Φ_ℓ, raising the power p in both sides of the inequality for some $p > 2$, taking the supremum over time and then taking expectation, and using the regularity results in **a)**, we get

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leq s\leq t} \Phi_\ell^p\big(u_{2n}(s), v_{2n}(s)\big)\Big] \\ &\leq C + 3^{p-1}\mathbb{E}\Big[\Phi_\ell^p\big(u_{2n}(0), v_{2n}(0)\big)\Big] + 3^{p-1}\int_0^t \mathbb{E}\Big[\sup_{0\leq r\leq s} \Phi_\ell^p\big(u_{2n}(r), v_{2n}(r)\big)\Big]\,\mathrm{d}r \\ &\quad + 3^{p-1}\mathbb{E}\Big[\sup_{0\leq s\leq t}\Big|\int_0^s \Big(\Delta_n^\ell v_n(r), \Delta_n^\ell\sigma_n\big(u_n(r), v_n(r)\big)\,\mathrm{d}W(r)\Big)\Big|^p\Big]. \end{aligned} \tag{A.7}$$

Using the Burkholder-Davis-Gundy inequality and previous estimates for $\ell = 0, \frac{1}{2}$, and using the regularity results in **a)**, we obtain

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leq s\leq t}\Big|\int_0^s \Big(\Delta_n^\ell v_{2n}(r), \Delta_n^\ell\sigma_n\big(u_n(r), v_n(r)\big)\mathrm{d}W(r)\Big)\Big|^p\Big] \\ &\leq C\mathbb{E}\Big[\Big(\int_0^t \|\Delta_n^\ell v_{2n}(s)\|_{\mathbb{L}^2}^2 \big\|\Delta_n^\ell\sigma_n\big(u_n(s), v_n(s)\big)\big\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big)^{p/2}\Big] \\ &\leq C + C\mathbb{E}\Big[\sup_{0\leq s\leq t}\Phi_\ell^p\big(u_{2n}(s), v_{2n}(s)\big)\Big] + C\int_0^t \mathbb{E}\Big[\sup_{0\leq s\leq t}\Phi_\ell^p\big(u_{2n}(s), v_{2n}(s)\big)\Big]\,\mathrm{d}s. \end{aligned} \tag{A.8}$$

Using (A.8) in (A.7) and using the Gronwall lemma we get for $\ell = 0, \frac{1}{2}$ and $p \geq 2$,

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}\Phi_\ell^p\big(u_{2n}(s), v_{2n}(s)\big)\Big] \leq C\,\mathbb{E}\Big[\Phi_\ell^p\big(u_{2n}(0), v_{2n}(0)\big)\Big]\,e^{CT} \leq C\,\mathbb{E}\Big[\Phi_\ell^p\big(u_0, v_0\big)\Big]\,e^{CT}. \tag{A.9}$$

b3) Dealing of cases $\ell = 1, \frac{3}{2}$. We assume **(A3)** for these two cases. If we treat $\sigma_2(v)$ and $F_2(v)$ as general functions, then the chain rule and the product rule formula of calculus will lead us to higher order derivative terms with higher moments in v in the right-hand side as compared to the left-hand side; see (A.10) and (A.11) below for the similar estimates in v. Then, the Gronwall lemma may not be applied. Thus, $F \equiv F(u)$ and $\sigma \equiv \sigma(u)$ are treated as general functions, but $F \equiv F(v)$ and $\sigma \equiv \sigma(v)$ are assumed to be only affine in v.

Case-1: Let us consider the case $\sigma \equiv \sigma_1(u)$ and $\widehat{F} \equiv F_1(u)$, which can be dealt as general functions. Take $\ell = 1$ in (A.5). Then, using product formula, and chain rule for general functions and by **(A4)** for $m = 1, 2$, we infer

$$\|\Delta_n\sigma_n(u_n)\|_{\mathbb{L}^2}^2 = \sum_{j=1}^n \lambda_j^2\big|\big(\sigma(u_n), \rho_j\big)\big|^2 \leq \|\Delta\sigma(u_n)\|_{\mathbb{L}^2}^2 \leq \tilde{C}_g^2\,\|(\nabla u_n)^2\|_{\mathbb{L}^2}^2 + C_g\|\Delta u_n\|_{\mathbb{L}^2}^2. \tag{A.10}$$

Now, using Ladyzhenskaya inequality and Poincaré inequality, we estimate the term

$$\begin{aligned}\|(\nabla u_n)^2\|_{\mathbb{L}^2}^2 &\leq C\|\nabla u_n\|_{\mathbb{L}^2}^2\|\Delta u_n\|_{\mathbb{L}^2}^2 \leq C\|\nabla u_n\|_{\mathbb{L}^2}^4\|\Delta u_n\|_{\mathbb{L}^2}^2 + C\|\Delta u_n\|_{\mathbb{L}^2}^2 \\ &\leq C\|\nabla u_n\|_{\mathbb{L}^2}^8 + C\|\Delta u_n\|_{\mathbb{L}^2}^4 + C\|\nabla\Delta u_n\|_{\mathbb{L}^2}^2. \end{aligned} \tag{A.11}$$

Similar estimates will hold for $\|\Delta_n \widehat{F}_n(u_n)\|^2_{\mathbb{L}^2}$. Now, we take $\ell = \frac{3}{2}$ in (A.5). Using the chain rule, **(A4)** for $m = 1, 2, 3$, we infer

$$\|\Delta_n^{3/2}\sigma_n(u_n)\|^2_{\mathbb{L}^2} \leq \|\Delta^{3/2}\sigma(u_n)\|^2_{\mathbb{L}^2} \leq \bar{C}_g^2 \, \|(\nabla u_n)^3\|^2_{\mathbb{L}^2} + \tilde{C}_g^2 \, \|\nabla u_n \Delta u_n\|^2_{\mathbb{L}^2} + C_g \|\nabla \Delta u_n\|^2_{\mathbb{L}^2} \,. \tag{A.12}$$

Using the Sobolev embeddings we further esimate

$$\|(\nabla u_n)^3\|^2_{\mathbb{L}^2} \leq C \, \|\nabla u_n\|^6_{\mathbb{L}^6} \leq C \, \|\Delta u_n\|^6_{\mathbb{L}^2} \,, \tag{A.13}$$

and

$$\begin{aligned} \|\nabla u_n \Delta u_n\|^2_{\mathbb{L}^2} &\leq C \, \|\nabla u_n\|^2_{\mathbb{L}^4} \|\Delta u_n\|^2_{\mathbb{L}^4} \leq C \, \|\nabla u_n\|^{1/2}_{\mathbb{L}^2} \|\Delta u_n\|^{3/2}_{\mathbb{L}^2} \|\Delta u_n\|^{1/2}_{\mathbb{L}^2} \|\nabla \Delta u_n\|^{3/2}_{\mathbb{L}^2} \\ &\leq C \|\nabla u_n\|^2_{\mathbb{L}^2} + C \|\Delta u_n\|^8_{\mathbb{L}^2} + C \|\nabla \Delta u_n\|^3_{\mathbb{L}^2} \,. \end{aligned} \tag{A.14}$$

Similar estimates will hold for $\|\Delta_n^{3/2} \widehat{F}_n(u_n)\|^2_{\mathbb{L}^2}$.

Case-2: Let $\sigma \equiv \sigma_2(v)$ and $\widehat{F} \equiv F_2(v)$, such that **(A5)** holds. For $\ell = 1$ we have

$$\|\Delta_n \sigma_n(v_n)\|^2_{\mathbb{L}^2} = \sum_{j=1}^{n} \lambda_j^2 \big|\big(\sigma(v_n), \rho_j\big)\big|^2 \leq \|\Delta \sigma(v_n)\|^2_{\mathbb{L}^2} \leq C_g \|\Delta v_n\|^2_{\mathbb{L}^2} \,, \tag{A.15}$$

and for $\ell = \frac{3}{2}$ we have

$$\|\Delta_n^{3/2} \sigma_n(v_n)\|^2_{\mathbb{L}^2} \leq \|\Delta^{3/2} \sigma(v_n)\|^2_{\mathbb{L}^2} \leq C_g \|\nabla \Delta v_n\|^2_{\mathbb{L}^2} \,. \tag{A.16}$$

Similar estimates will hold for $\|\Delta_n^{3/2} \widehat{F}_n(v_n)\|^2_{\mathbb{L}^2}$.

Using the estimates (A.10), (A.14), (A.15), and (A.16) in (A.5) for $\ell = 1, \frac{3}{2}$, and using the regularity results proved so far for u_{1n} and u_{2n} and their time derivatives, we get (A.7) for $\ell = 1, \frac{3}{2}$. Finally, the use of Burkholder-Davis-Gundy inequality yields the assertion for $\ell = 1, \frac{3}{2}$.

2) Convergence: By step **1)**, for $p \geq 2$

$$(u_{2n}, v_{2n})_n \subset L^p\big(\Omega; L^\infty(0, T; \mathbb{H}^{2\ell+1} \times \mathbb{H}^{2\ell})\big) \cap L^p\big(\Omega; L^2(0, T; \mathbb{H}^{2\ell+1} \times \mathbb{H}^{2\ell})\big)$$

is bounded for $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}$. Here, we need to argue the convergence case by case. First, consider $\ell = 0$. Then, there exist subsequences $(u_{2n'})_{n'}$ and $(v_{2n'})_{n'}$, which converge weakly to u_2' and v_2', respectively. Then, using standard arguments (see [3]) shows that (u_2', v_2') is a weak solution of (A.2). By the uniqueness of the weak solution, we have $(u_2', v_2') = (u, v)$. By Fatou's lemma, passing to the limit in (A.9) yields

$$\mathbb{E}\bigg[\sup_{0 \leq s \leq t} \Phi_\ell^p\big(u_2(s), v_2(s)\big)\bigg] \leq C \mathbb{E}\left[\Phi_\ell^p\big(u_0, v_0\big)\right] e^{CT} \,, \tag{A.17}$$

for $\ell = 0$. Now, consider $\ell = \frac{1}{2}$. Then, there exist subsequences $(u_{2n''})_{n''}$ and $(v_{2n''})_{n''}$ which converge weakly to some $\tilde{u}_2$ and $\tilde{v}_2$, respectively. By using the standard arguments and the uniqueness of the solution of the system (A.2), we claim that $(\tilde{u}_2, \tilde{v}_2) = (\nabla u_2, \nabla v_2)$. Thus, by passing to the limit (A.17) holds for $\ell = 1/2$. Similar arguments will yield the result for $\ell = 1, \frac{3}{2}$. Combining (A.17) with (A.3) we get the assertions in Lemma 3.2. □

Appendix B. Proof of Hölder continuity in time

The proof of Lemma 3.3 uses the regularity results for the variational solution of (3.1)–(3.2) in Lemma 3.2. We obtain a Hölder regularity in time for u which is double the one for v: the reason for it is the occurrence of the Itô integral in (3.2), but not in (3.1).

Proof of Lemma 3.3. **Proof of** (*i*)**.** Let $r, s \in [0,T]$, and fix $p \in \mathbb{N}$. By Lemma 3.2 (*i*), we have $v \in L^2\big(\Omega; L^\infty(0,T;\mathbb{L}^2)\big)$. Therefore, $\int_s^r v(\xi)\mathrm{d}\xi$ is well-defined for *a.e.* $x \in \mathcal{O}$ and $\mathbb{P}$-a.s.. Thus, we can write the weak formulation (3.1) in strong form $\mathbb{P}$-a.s. as

$$u(r) - u(s) = \int_s^r v(\xi)\mathrm{d}\xi, \quad \text{for a.e. } x \in \mathcal{O}, \text{ for } r, s \in [0,T]\,.$$

Then, the Hölder inequality yields

$$\|u(r) - u(s)\|_{\mathbb{L}^2}^{2p} \le \Big(\int_s^r \|v(\xi)\|_{\mathbb{L}^2}\,\mathrm{d}\xi \Big)^{2p} \le |r-s|^{2p-1} \int_s^r \|v(\xi)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}\xi\,.$$

We fix $s, t \in [0,T]$, and take supremum w.r.t. r, and then take expectation to get

$$\mathbb{E}\Big[\sup_{s\le r\le t} \|u(r) - u(s)\|_{\mathbb{L}^2}^{2p}\Big] \le |t-s|^{2p-1}\,\mathbb{E}\Big[\int_s^t \|v(\xi)\|_{\mathbb{L}^2}^{2p}\,\mathrm{d}\xi\Big] \le |t-s|^{2p}\,\mathbb{E}\Big[\sup_{0\le t\le T} \|v(t)\|_{\mathbb{L}^2}^{2p}\Big]\,.$$

Hence, (*i*) holds by applying (3.3) in Lemma 3.2.

Proof of (*ii*)**.** Let $r, s \in [0,T]$, and fix $p \in \mathbb{N}$. The first part follows as (*i*). By Lemma 3.2 (*ii*), we have $u \in L^2\big(\Omega; L^\infty(0,T;\mathbb{H}^2)\big)$. Therefore, $\int_s^r \Delta u(\xi)\,\mathrm{d}\xi$ is well-defined for *a.e.* $x \in \mathcal{O}$ and $\mathbb{P}$-a.s.. By Lemma 3.2 (*i*), we have $(u,v) \in L^2\big(\Omega; L^\infty(0,T;\mathbb{H}^2 \times \mathbb{H}^1)\big)$. Therefore, by **(A3)**, $\int_s^r F\big(u(\xi), v(\xi)\big)\mathrm{d}\xi$ is well-defined for *a.e.* $x \in \mathcal{O}$ for $s, r \in [0,T]$ and $\mathbb{P}$-a.s.. Similarly, by **(A3)** and Itô isometry, $\int_s^r \sigma\big(u(\xi), v(\xi)\big)\mathrm{d}W(\xi)$ is well-defined for *a.e.* $x \in \mathcal{O}$ for $s, r \in [0,T]$ and $\mathbb{P}$-a.s.. Now, from the weak formulation (3.2) and using the above conclusion, we may rewrite the equation in the strong form as (see [8, Section 6.3, Remark (ii)])

$$v(r) - v(s) = \int_s^r \Delta u(\xi)\mathrm{d}\xi + \int_s^r F\big(u(\xi), v(\xi)\big)\mathrm{d}\xi + \int_s^r \sigma\big(u(\xi), v(\xi)\big)\mathrm{d}W(\xi). \tag{B.1}$$

By Hölder inequality, we estimate

$$\begin{aligned} \|v(r) - v(s)\|_{\mathbb{L}^2}^2 &\le C(r-s)\int_s^r \|\Delta u(\xi)\|_{\mathbb{L}^2}^2\,\mathrm{d}\xi + C(r-s)\int_s^r \big\|F\big(u(\xi), v(\xi)\big)\big\|_{\mathbb{L}^2}^2\,\mathrm{d}\xi \\ &\quad + C\int_{\mathcal{O}} \Big(\int_s^r \sigma\big(u(\xi), v(\xi)\big)\,\mathrm{d}W(\xi)\Big)^2\,\mathrm{d}x\,. \end{aligned} \tag{B.2}$$

We fix $s, t \in [0,T]$, and take supremum w.r.t. r, then take expectation. Using **(A3)**, Itô isometry, Lemma 3.2 (*i*) and (*ii*), we infer

$$\begin{aligned} \mathbb{E}\Big[\sup_{s\le r\le t} \|v(r) - v(s)\|_{\mathbb{L}^2}^2\Big] &\le C\mathbb{E}\Big[\sup_{0\le t\le T} \|\Delta u(t)\|_{\mathbb{L}^2}^2\Big](t-s)^2 + C(t-s)^2 + C\mathbb{E}\Big[\sup_{0\le t\le T}\|\nabla u(t)\|_{\mathbb{L}^2}^2\Big](t-s)^2 \\ &\quad + C\mathbb{E}\Big[\sup_{0\le t\le T}\|v(t)\|_{\mathbb{L}^2}^2\Big](t-s)^2 + C\mathbb{E}\Big[\int_s^t \big(1 + \|\nabla u(\xi)\|_{\mathbb{L}^2}^2 + \|v(\xi)\|_{\mathbb{L}^2}^2\big)\Big] \\ &\le C(t-s)\,. \end{aligned}$$

Proof of (*iii*)**.** Let $r, s \in [0,T]$, and fix $p \in \mathbb{N}$. The first part follows as (*i*). In order to verify the bound for $\mathbb{E}\big[\sup_{s\le r\le t}\|\nabla[v(r) - v(s)]\|_{\mathbb{L}^2}^2\big]$, consider the equation (B.1). By Lemma 3.2, we have $(u,v) \in L^{2p}\big(\Omega; L^\infty(0,T;\mathbb{H}^4 \times \mathbb{H}^3)\big)$. Thus, by **(A3)**, **(A4)**, we can take the gradients in (B.1), since it is a closed operator on $\mathbb{H}^1$. Then, the terms are well-defined. Proceeding similarly as part (ii) we get

$$\mathbb{E}\Big[\sup_{s\leq r\leq t}\|\nabla[v(r)-v(s)]\|_{\mathbb{L}^2}^2\Big] \leq C\mathbb{E}\Big[\int_s^t \|\nabla\Delta u(\xi)\|_{\mathbb{L}^2}^2\,\mathrm{d}\xi\Big](t-s) + C\mathbb{E}\Big[\int_s^t \big\|\nabla F\big(u(\xi),v(\xi)\big)\big\|_{\mathbb{L}^2}^2\,\mathrm{d}\xi\Big](t-s)$$
$$+ C\mathbb{E}\Big[\int_s^t \big\|\nabla\sigma\big(u(\xi),v(\xi)\big)\big\|_{\mathbb{L}^2}^2\,\mathrm{d}\xi\Big].$$

By **(A4)**, $\big\|\nabla\sigma\big(u(\xi),v(\xi)\big)\big\|_{\mathbb{L}^2}^2 \leq C_g^2\big(\|\nabla u(\xi)\|_{\mathbb{L}^2}^2 + \|\nabla v(\xi)\|_{\mathbb{L}^2}^2\big)$. Then, using Lemma 3.2 (i), (ii) and (iii), we further estimate

$$\leq C(t-s)^2 + C\mathbb{E}\Big[\int_s^t \big(\|\nabla u(\xi)\|_{\mathbb{L}^2}^2 + \|\nabla v(\xi)\|_{\mathbb{L}^2}^2\big)\Big] \leq C(t-s).$$

Proof of (iv)**.** Let $r,s\in[0,T]$, and fix $p\in\mathbb{N}$. The first part follows as (i). To verify the bound for $\mathbb{E}\big[\sup_{s\leq r\leq t}\|\Delta[v(r)-v(s)]\|_{\mathbb{L}^2}^2\big]$, consider (B.1). Argue similarly as part (ii) to apply the Laplacian to (B.1) due to **(A3)**, **(A4)**, and proceed similarly to obtain

$$\mathbb{E}\Big[\sup_{s\leq r\leq t}\|\Delta[v(r)-v(s)]\|_{\mathbb{L}^2}^2\Big] \leq C\mathbb{E}\Big[\int_s^t \|\Delta^2 u(\xi)\|_{\mathbb{L}^2}^2\,\mathrm{d}\xi\Big](t-s) + C\mathbb{E}\Big[\int_s^t \big\|\Delta F\big(u(\xi),v(\xi)\big)\big\|_{\mathbb{L}^2}^2\,\mathrm{d}\xi\Big](t-s)$$
$$+ C\mathbb{E}\Big[\int_s^t \big\|\Delta\sigma\big(u(\xi),v(\xi)\big)\big\|_{\mathbb{L}^2}^2\,\mathrm{d}\xi\Big].$$

To bound the last two terms requires **(A3)** to *e.g.* write $\sigma\big(u(\xi),v(\xi)\big) = \sigma_1\big(u(\xi)\big) + \sigma_2\big(v(\xi)\big)$, where σ_2 is affine in v. Then, $\Delta\sigma\big(u(\xi),v(\xi)\big) = \Delta\sigma_1\big(u(\xi)\big)$ and we can follow the steps of (A.10)-(A.14) to bound it. Similar techniques may be used to deal with $\|\Delta F\big(u(\xi),v(\xi)\big)\|_{\mathbb{L}^2}^2$. Lemma 3.2 then settles the assertion. Thus, the proof is complete. □

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