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Continuous dependence estimate for conservation laws with Lévy noise

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Abstract

We are concerned with multidimensional stochastic balance laws driven by Lévy processes. Using bounded variation (BV) estimates for vanishing viscosity approximations, we derive an explicit continuous dependence estimate on the nonlinearities of the entropy solutions under the assumption that Lévy noise only depends on the solution. This result is used to show the error estimate for the stochastic vanishing viscosity method. In addition, we establish fractional BV estimate for vanishing viscosity approximations in case the noise coefficient depends on both the solution and spatial variable.

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1. Introduction

In this paper, we derive continuous dependence estimate based on nonlinearities for stochastic conservation laws driven by multiplicative Lévy noise. Our problem of interest is a stochastic partial differential equation (SPDE) and it is defined on a filtered probability space

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$\big(\Omega, P, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}\big)$, where the unknown $u(t,\cdot)$ is an $L^p(\mathbb{R}^d)$-valued predictable process that satisfies the Cauchy problem

$$\begin{cases} du(t,x) + \mathrm{div}_x F(u(t,x))\,dt = \int_{|z|>0} \eta(u(t,x);z)\,\tilde{N}(dz,dt), & x \in \Pi_T, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $\Pi_T = \mathbb{R}^d \times (0,T)$ with $T > 0$ fixed. The initial data $u_0(x)$ is a given function on $\mathbb{R}^d$, and $F : \mathbb{R} \mapsto \mathbb{R}^d$ is a given (sufficiently smooth) vector valued flux function (see Section 2 for the complete list of assumptions). The right hand side of (1.1) signifies the Lévy noise term and it is represented by a compensated Poisson random measure $\tilde{N}(dz,dt) = N(dz,dt) - \nu(dz)\,dt$, where N is a Poisson random measure on $\mathbb{R} \times (0,\infty)$ with intensity measure $\nu(dz)$. The integrand $\eta(u,z)$ is a real valued function.

In the case $\eta = 0$, the problem (1.1) becomes a standard conservation law in $\mathbb{R}^d$ and its well-posedness analysis has a very long tradition, going back to the 1950s. The question of existence and uniqueness of solutions of conservation laws was first settled in the pioneering papers of Kružkov [15] and Vol'pert [17]. For a completely satisfactory well-posedness theory of conservation laws, we refer to the monograph of Dafermos [8]. See also [12] and references therein.

Evolutionary SPDEs with Lévy noise has been the topic of interest of many authors lately, and new results are emerging faster than ever before. However, until recently the study of balance laws driven by noise was largely limited to problems with Brownian noise. The interest in problems with Lévy noise is rather recent and this article is a part of this developing story.

1.1. Stochastic balance laws driven by Brownian white noise

One can safely say that there is a satisfactory well-posedness theory for problems with Brownian noise. If the noise is of additive nature, a change of variable reduces the equation into a hyperbolic conservation law with random flux which could be analyzed with deterministic techniques (cf. [14]) à la Kružkov.

The case of multiplicative noise is more subtle, one could not apply a straightforward Kružkov's doubling method to get uniqueness. The first breakthrough on this topic was by Feng and Nualart [11], who established uniqueness of entropy solution by recovering additional information from the vanishing viscosity method. The existence was proven using stochastic version of compensated compactness and it was valid for *one* spatial dimension. A number of authors have contributed since then, and we mention the works of Debussche and Vovelle [9], Chen et al. [6], Bauzet et al. [1] and Biswas and Majee [3]. We want to specifically mention the work [6] of Chen et al., where well-posedness of entropy solution is established in $L^p \cap BV$, via BV framework. More importantly, the BV framework enables the authors to derive continuous dependence estimate and, as a by product, one gets an explicit *convergence rate* for vanishing viscosity method.

1.2. Stochastic balance laws driven by Lévy noise

There is a large body of literature (see the book [16] and references therein) on SPDEs driven by Lévy noise, but the available theory is not general enough to cover (1.1). Roughly speaking, the theory developed in [16] covers quasi-linear parabolic equations driven by Lévy noise and typically the solutions of such equations enjoy regularizing effect. A comprehensive entropy solution theory, within L^p-solution framework, for (1.1) is made available by Biswas et al. [2]

very recently. We also mention that Dong and Xu [10] established the global well-posedness of strong, weak and mild solutions for one-dimensional viscous Burger's equation driven by Poisson process with Dirichlet boundary condition.

Irrespective of the smoothness of the initial data $u_0(x)$, due to the presence of nonlinear flux term in equation (1.1), solutions to (1.1) are not necessarily smooth and weak solutions must be sought. Before introducing the concept of weak solutions, we first assume that the filtered probability space $\big(\Omega, P, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}\big)$ satisfies the usual hypothesis. The notion of weak solution is defined as follows:

Definition 1.1 *(Weak solution).* An $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t : t \ge 0\}$-predictable stochastic process $u(t) = u(t,x)$ is called a stochastic weak solution of (1.1) if for all non-negative test functions $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \psi(0,x)u(0,x)\,dx + \int_{\mathbb{R}^d}\int_0^T \Big\{\partial_t\psi(t,x)u(t,x) + F(u(t,x))\cdot\nabla_x\psi(t,x)\Big\}\,dx\,dt$$

$$+\int_{t=0}^T \int_{|z|>0}\int_{\mathbb{R}^d} \eta(u(t,x);z)\psi(t,x)\,dx\,\tilde{N}(dz,dt) = 0, \quad P\text{-a.s.} \tag{1.2}$$

However, it is well known that weak solutions may be discontinuous and they are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. The notion of entropy solution requires introduction of entropy–entropy flux pairs.

Definition 1.2 *(Entropy–entropy fux pair).* An ordered pair (β,ζ) is called an entropy–entropy flux pair if $\beta \in C^2(\mathbb{R})$ with $\beta \ge 0$, and $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_d) : \mathbb{R} \mapsto \mathbb{R}^d$ is a vector field satisfying

$$\zeta'(r) = \beta'(r)F'(r), \text{ for all r.}$$

Moreover, an entropy–entropy flux pair (β,ζ) is called convex if $\beta''(\cdot) \ge 0$.

Now the notion of stochastic entropy solution could be defined as follows:

Definition 1.3 *(Stochastic entropy solution).* An $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t : t \ge 0\}$-predictable stochastic process $u(t) = u(t,x)$ is called a stochastic entropy solution of (1.1) provided

(1) For each $T > 0$, $p = 2, 3, 4, \cdots$,

$$\sup_{0\le t\le T} E\Big[||u(t,\cdot)||_p^p\Big] < \infty.$$

(2) For all test functions $0 \le \psi \in C_c^{1,2}([0,\infty)\times\mathbb{R}^d)$, and each convex entropy pair (β,ζ),

$$\int_{\mathbb{R}^d_x} \psi(0,x)\beta(u(0,x))\,dx + \int_{\Pi_T} \Big\{\partial_t\psi(t,x)\beta(u(t,x)) + \zeta(u(t,x))\cdot\nabla_x\psi(t,x)\Big\}\,dx\,dt$$

$$
\begin{aligned}
&+ \int_{r=0}^{T} \int_{|z|>0} \int_{\mathbb{R}^d_x} \Big(\beta\big(u(r,x) + \eta(u(r,x);z)\big) - \beta\big(u(r,x)\big)\Big) \psi(r,x)\, dx\, \tilde{N}(dz, dr) \\
&+ \int_{\Pi_T} \int_{|z|>0} \Big(\beta\big(u(r,x) + \eta(u(r,x);z)\big) - \beta(u(r,x)) \\
&- \eta(u(r,x);z)\beta'(u(r,x))\Big) \psi(r,x)\, \nu(dz)\, dr\, dx \\
&\quad \ge 0 \quad P\text{-a.s.}
\end{aligned}
$$

The work in [2] establishes existence and uniqueness of entropy solution for the multidimensional Cauchy problem (1.1).

1.3. Scope and outline of this paper

We are primarily motivated by [6] and wish to develop a continuous dependence theory for stochastic entropy solution framework for (1.1). The rate of convergence for vanishing viscosity approximation to (1.1) then follows easily. However, we use the technology from [6] and it requires a priori BV bounds for the entropy solutions and this could be ensured provided the initial data lies in $u_0 \in L^p(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$.

Finally, we turn our discussions to more general form of (1.1), namely when the function η has explicit dependency on the spatial position x as well. In view of results in [2], this problem has unique entropy solution and, following [6], we derive a fractional BV estimate.

The remaining part of this paper is organized as follows: we collect all the assumptions needed in the subsequent analysis, results for the regularized problem and finally state the main results in Section 2. In Section 3, we prove uniform spatial BV estimate for the solution of vanishing viscosity approximation of (1.1), and thereby establishing BV bounds for entropy solutions. Section 4 deals with the continuous dependence estimate, while Section 5 deals with the error estimate. Finally, in Section 6, we establish a fractional BV estimate for a larger class of stochastic balance laws.

2. Preliminaries

Throughout this paper we use C, K to denote generic constants; the actual values of C, K may change from one line to the next during a calculation. The Euclidean norm on any $\mathbb{R}^d$-type space is denoted by $|\cdot|$ and the semi-norm in $\mathrm{BV}(\mathbb{R}^d)$ is denoted by $|\cdot|_{BV(\mathbb{R}^d)}$.

Next, we collect all the basic assumptions on the data of the problem (1.1).

(**A**.1) The initial data $u_0(x)$ is a $\cap_{p=1,2,..} L^p(\mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variable satisfying

$$
E\Big[||u_0||_p^p + ||u_0||_2^p + |u_0|_{BV(\mathbb{R}^d)}\Big] < \infty, \qquad \text{for } p = 1, 2, \ldots.
$$

(**A**.2) For every $k = 1, 2, \ldots, d$, the functions $F_k(s) \in C^2(\mathbb{R})$, and $F_k(s)$, $F_k'(s)$ and $F_k''(s)$ have at most polynomial growth in s.

(**A**.3) There exist positive constants $0 \le \lambda^* < 1$ and $C > 0$, such that for all $u, v \in \mathbb{R}$; $z \in \mathbb{R}$

$$|\eta(u;z) - \eta(v;z)| \le \lambda^* |u - v|(|z| \wedge 1), \quad \text{and} \quad |\eta(u;z)| \le C(1 + |u|)(|z| \wedge 1).$$

(**A**.4) The Lévy measure $\nu(dz)$ which has a possible singularity at $z = 0$, satisfies

$$\int_{|z|>0} (1 \wedge |z|^2)\, \nu(dz) < +\infty.$$

Remark 2.1. Note that we need the assumption (**A**.2) as a result of the requirement that the entropy solutions satisfy L^p bounds for all $p \ge 2$, which in turn forces us to choose initial data satisfying (**A**.1). However, it is possible to get entropy solution for initial data in $L^2(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, provided the given flux function is globally Lipschitz. The assumptions (**A**.1)–(**A**.4) are natural in view of [2] and they collectively ensure existence and uniqueness of stochastic entropy solution of (1.1).

To this end, for any given fixed $\epsilon > 0$, we consider the viscous perturbation of (1.1)

$$\begin{aligned} du_\epsilon(t,x) + \operatorname{div}_x F_\epsilon(u_\epsilon(t,x))\, dt &= \int_{|z|>0} \eta_\epsilon(u_\epsilon(t,x); z)\, \tilde{N}(dz, dt) + \epsilon \Delta_{xx} u_\epsilon\, dt, \;\; t > 0, \;\; x \in \mathbb{R}^d, \\ u(0,x) &= u_\epsilon(0,x), \;\; x \in \mathbb{R}^d, \end{aligned} \tag{2.1}$$

where $u_\epsilon(0,x)$ is a smooth approximation of initial data $u_0(x) \in L^p \cap BV(\mathbb{R}^d)$ such that

$$\begin{aligned} & E\Big[\int_{\mathbb{R}^d_x} |u_\epsilon(0,x)|^p\, dx\Big] \le E\Big[\int_{\mathbb{R}^d_x} |u_0(x)|^p\, dx\Big] \\ \text{and} \quad & E\Big[\int_{\mathbb{R}^d_x} |\nabla u_\epsilon(0,x)|\, dx\Big] \le E\Big[\int_{\mathbb{R}^d_x} |\nabla u_0(x)|\, dx\Big]. \end{aligned}$$

Let $F_\epsilon, \eta_\epsilon$ be "sufficiently smooth" approximations of F and η respectively, defined as in [2, Subsection 3.2]. Then F_ϵ and η_ϵ satisfy the same properties as F and η respectively (cf. (**A**.2)–(**A**.3)) and

$$\begin{aligned} |F_\epsilon(r) - F(r)| &\le C\epsilon(1 + |r|^{p_0}), \text{ for some } p_0 \in \mathbb{N}, \\ |\eta_\epsilon(u;z) - \eta(u;z)| &\le C\epsilon(1 + |u|)(1 \wedge |z|). \end{aligned} \tag{2.2}$$

The existence of global smooth solutions for (2.1) is detailed in [2], and the following proposition holds.

Proposition 2.1. *Let the assumptions* (**A**.1)–(**A**.4) *hold and* $\epsilon > 0$ *be given. Then there exists a unique* $C^2(\mathbb{R}^d)$*-valued predictable process* $u_\epsilon(t,\cdot)$ *which solves the initial value problem* (2.1). *Moreover,* (*a*) *for positive integers* $p = 2, 3, \cdots$, *and* $T > 0$

$$\sup_{\epsilon > 0} \sup_{0 \le t \le T} E\Big[||u_\epsilon(t, \cdot)||_p^p \Big] < \infty. \tag{2.3}$$

(*b*) *For a function* $\beta \in C^2(\mathbb{R})$ *with* β, β', β'' *having at most polynomial growth,*

$$\sup_{\epsilon > 0} E\left[\left| \epsilon \int_{t=0}^{T} \int_{\mathbb{R}_x^d} \beta''(u_\epsilon(t, x)) |\nabla_x u_\epsilon(t, x)|^2 \, dx \, dt \right|^p \right] < \infty, \quad p = 1, 2, \ldots, ; \ T > 0.$$

Remark 2.2. In view of Proposition 2.1 and assumption (**A**.1), it follows that, for each fixed $\epsilon > 0$, $\nabla u_\epsilon(t, x)$ is integrable. Moreover if $E\Big[\int_{\mathbb{R}_x^d} |\nabla^2 u_\epsilon(0, x)| \, dx \Big] < +\infty$, then $\nabla^2 u_\epsilon(t, x)$ is also integrable for fixed $\epsilon > 0$ and any finite time $T > 0$ (cf. [2, Section 3]).

Now we are in a position to state the main results of this article.

Main Theorem *(Continuous dependence estimate). Let the assumptions* (**A**.1), (**A**.2), (**A**.3), *and* (**A**.4) *hold for two given sets of data* (u_0, F, η) *and* (v_0, G, σ). *Let* $u(t, x)$ *be any entropy solution of* (1.1) *with initial data* $u_0(x)$ *and* $v(s, y)$ *be another entropy solution with initial data* $v_0(y)$ *and satisfies*

$$dv(s, y) + div_y G(v(s, y)) \, ds = \int_{|z|>0} \sigma(v(s, y); z) \, \tilde{N}(dz, ds). \tag{2.4}$$

In addition, we assume that F'', $F' - G' \in L^\infty$ *and define*

$$\mathcal{D}(\eta, \sigma) := \sup_{u \in \mathbb{R}} \int_{|z|>0} \frac{\big(\eta(u; z) - \sigma(u; z)\big)^2}{1 + |u|^2} \nu(dz).$$

Then there exists a constant $C_T > 0$, *independent of* $|u_0|_{BV(\mathbb{R}^d)}$ *and* $|v_0|_{BV(\mathbb{R}^d)}$, *such that for a.e.* $t \ge 0$,

$$\begin{aligned}
&E\Big[\int_{\mathbb{R}_x^d} \big| u(t, x) - v(t, x) \big| \phi(x) \, dx \Big] \\
&\le C_T \Bigg[\Big(1 + E[|v_0|_{BV(\mathbb{R}^d)}]\Big) \sqrt{t \mathcal{D}(\eta, \sigma)} ||\phi(\cdot)||_{L^\infty(\mathbb{R}^d)} \\
&\quad + E\Big[|v_0|_{BV(\mathbb{R}^d)}\Big] \, ||F' - G'||_\infty \, t \, ||\phi(\cdot)||_{L^\infty(\mathbb{R}^d)} \\
&\quad + E\Big[\int_{\mathbb{R}_x^d} |u_0(x) - v_0(x)| \phi(x) \, dx \Big] + \sqrt{t \mathcal{D}(\eta, \sigma)} ||\phi(\cdot)||_{L^1(\mathbb{R}^d)} \Bigg],
\end{aligned} \tag{2.5}$$

where $0 \le \phi \in C_c^2(\mathbb{R}^d)$ *such that* $|\nabla\phi(x)| \le C\phi(x)$ *and* $|\Delta\phi(x)| \le C\phi(x)$ *for some constant* $C > 0$*. Moreover, a special choice of* $\phi(x)$ *with the above properties*

$$\phi(x) = \begin{cases} 1, & \text{when } |x| \le R, \\ e^{-C(|x|-R)}, & \text{when } |x| \ge R, \end{cases}$$

leads to the following simplified result: for any $R > 0$*, there exists a constant* $C_T^R > 0$*, independent of* $|u_0|_{BV(\mathbb{R}^d)}$ *and* $|v_0|_{BV(\mathbb{R}^d)}$*, such that for a.e.* $t \ge 0$*,*

$$\begin{aligned} E\Big[\int\limits_{|x|\le R} \big|u(t,x) - v(t,x)\big|\,dx\Big] \\ \le C_T^R \Bigg[&\big(1 + E[|v_0|_{BV(\mathbb{R}^d)}]\big)\sqrt{t\mathcal{D}(\eta,\sigma)} + t\,E\big[|v_0|_{BV(\mathbb{R}^d)}\big]\,||F' - G'||_\infty \\ &+ E\Big[\int\limits_{\mathbb{R}_x^d} |u_0(x) - v_0(x)|\,dx\Big]\Bigg]. \end{aligned} \tag{2.6}$$

Remark 2.3. The conditions that F'', $F' - G' \in L^\infty$ could be avoided if we assume that $u, v \in L^\infty((0,T) \times \mathbb{R}^d \times \Omega)$ for any time $T > 0$. In that case, an appropriate version of the main theorem would be possible. Moreover, the quantity $\mathcal{D}(\eta, \sigma)$ is well defined in view of (**A**.3) and (**A**.4).

As a by product of the above theorem, we have the following corollary:

Main Corollary *(Error estimate). Let the assumptions* (**A**.1), (**A**.2), (**A**.3)*, and* (**A**.4) *hold and let* $u(t,x)$ *be any entropy solution of* (1.1) *with* $E\big[|u(t,\cdot)|_{BV(\mathbb{R}^d)}\big] \le E\big[|u_0|_{BV(\mathbb{R}^d)}\big]$*, for* $t > 0$*. In addition, we assume that* $F'' \in L^\infty$*. Then, there exists a constant* $C_T > 0$*, independent of* $|u_0|_{BV(\mathbb{R}^d)}$*, such that for a.e.* $t \ge 0$

$$\begin{aligned} E\Big[\int\limits_{\mathbb{R}_x^d} \big|u_\epsilon(t,x) - u(t,x)\big|\,dx\Big] \le C_T \Big\{ &\epsilon^{\frac{1}{2}}\big(1 + E[|u_0|_{BV(\mathbb{R}^d)}]\big)(1+t) \\ &+ E\Big[\int\limits_{\mathbb{R}_x^d} \big|u_\epsilon(0,x) - u_0(x)\big|\,dx\Big]\Big\}. \end{aligned}$$

Therefore, if the initial error $E\big[\int_{\mathbb{R}_x^d} \big|u_\epsilon(0,x) - u_0(x)\big|\,dx\big] \le C\epsilon^{\frac{1}{2}}$, then $E\big[\int_{\mathbb{R}_x^d} \big|u_\epsilon(t,x) - u(t,x)\big|\,dx\big] \le C\epsilon^{\frac{1}{2}}$. In other words, the convergence rate is the same as deterministic problem and hence optimal.

We finish this section by recalling a special class of entropy functions $\big(\beta_\xi(r)\big)_{\xi>0}$, as described in [2, Section 2], satisfying

$$|r| - M_1\xi \le \beta_\xi(r) \le |r| \quad \text{and} \quad |\beta_\xi''(r)| \le \frac{M_2}{\xi}\mathbb{1}_{\{|r|\le\xi\}}, \tag{2.7}$$

where $M_1 = \sup_{|r|\le 1} \big||r| - \beta_1(r)\big|$, and $M_2 = \sup_{|r|\le 1} |\beta_1''(r)|$. Finally, by dropping ξ, for $\beta = \beta_\xi$ we define

$$F_k^\beta(a,b) = \int_b^a \beta'(\sigma - b) F_k'(\sigma)\, d(\sigma), \quad F^\beta(a,b) = (F_1^\beta(a,b), F_2^\beta(a,b), \dots, F_d^\beta(a,b)),$$

$$F_k(a,b) = \operatorname{sign}(a-b)(F_k(a) - F_k(b)), \quad F(a,b) = (F_1(a,b), F_2(a,b), \dots, F_d(a,b)).$$

3. A priori estimates

In this section, we derive uniform spatial BV bound for the stochastic balance laws driven by Lévy process given by (1.1) under the assumptions (**A**.1), (**A**.2), (**A**.3), and (**A**.4).

Theorem 3.1 *(Spatial bounded variation). Let the assumptions* (**A**.1), (**A**.2), (**A**.3)*, and* (**A**.4) *hold. Furthermore, let $u_\epsilon(t,x)$ be a solution to the initial value problem (2.1). Then, for any time $t > 0$*

$$E\Big[\int_{\mathbb{R}_x^d} \big|\nabla u_\epsilon(t,x)\big|\,dx\Big] \le E\Big[\int_{\mathbb{R}_x^d} \big|\nabla u_\epsilon(0,x)\big|\,dx\Big] \le E\Big[\int_{\mathbb{R}_x^d} \big|\nabla u_0(x)\big|\,dx\Big].$$

Proof. Since $u_\epsilon(t,x)$ is a smooth solution of the initial value problem (2.1), by differentiating (2.1) with respect to x_i, we find that $\partial_{x_i} u_\epsilon(t,x)$ satisfies the stochastic partial differential equation given by

$$d\big(\partial_{x_i} u_\epsilon(t,x)\big) + \operatorname{div}_x\big(F_\epsilon'(u_\epsilon(t,x))\partial_{x_i} u_\epsilon(t,x)\big)\,dt = \int_{|z|>0} \eta_\epsilon'(u_\epsilon(t,x);z)\partial_{x_i} u_\epsilon(t,x)\tilde{N}(dz,dt) \\ + \epsilon\Delta_{xx}(\partial_{x_i} u_\epsilon(t,x))\,dt.$$

To proceed further, we apply Itô–Lévy formula to $\beta_\xi(\partial_{x_i} u_\epsilon(t,x))$ to obtain

$$\begin{aligned} &d\big(\beta_\xi(\partial_{x_i} u_\epsilon(t,x))\big) + \operatorname{div}_x\big(F_\epsilon'(u_\epsilon(t,x))\partial_{x_i} u_\epsilon(t,x)\big)\,\beta_\xi'(\partial_{x_i} u_\epsilon(t,x))\,dt \\ &= \int_{|z|>0}\int_{\theta=0}^{1} \eta_\epsilon'(u_\epsilon(t,x);z)\partial_{x_i} u_\epsilon(t,x)\beta_\xi'\big(\partial_{x_i} u_\epsilon(t,x) + \theta\,\eta_\epsilon'(u_\epsilon(t,x);z)\partial_{x_i} u_\epsilon(t,x)\big)\,d\theta\,\tilde{N}(dz,dt) \\ &\quad + \int_{|z|>0}\int_{\theta=0}^{1} (1-\theta)\big(\eta_\epsilon'(u_\epsilon;z)\partial_{x_i} u_\epsilon\big)^2 \beta_\xi''\Big(\partial_{x_i} u_\epsilon(t,x) + \theta\,\eta_\epsilon'(u_\epsilon(t,x);z)\partial_{x_i} u_\epsilon(t,x)\Big)\,d\theta\,\nu(dz)\,dt \\ &\quad + \epsilon\Delta_{xx}\big(\partial_{x_i} u_\epsilon(t,x)\big)\beta_\xi'(\partial_{x_i} u_\epsilon(t,x))\,dt. \end{aligned} \tag{3.1}$$

Since β_ξ is convex, we conclude that $\epsilon\Delta_{xx}\big(\partial_{x_i} u_\epsilon(t,x)\big)\beta_\xi'\big(\partial_{x_i} u_\epsilon(t,x)\big) \le \epsilon\Delta\beta_\xi\big(\partial_{x_i} u_\epsilon(t,x)\big)$ and the martingale term has zero expectation. Moreover, by Remark 2.2, we see that for each fixed $\epsilon > 0$ and $1 \le i \le d$, $\nabla\partial_{x_i} u_\epsilon(t,x)$ is integrable. Let $0 \le \psi(x) \in C_c^\infty(\mathbb{R}^d)$. Multiply (3.1) by ψ and then integrate with respect to x we have

$$
\begin{aligned}
E\Big[\int_{\mathbb{R}^d_x} \beta_\xi\big(\partial_{x_i} u_\epsilon(t,x)\big)\psi(x)\,dx\Big] &\le E\Big[\int_{\mathbb{R}^d_x} \beta_\xi\big(\partial_{x_i} u_\epsilon(0,x)\big)\psi(x)\,dx\Big] \\
&+ E\Big[\int_{\mathbb{R}^d_x}\int_{s=0}^{t}\int_{|z|>0}\int_{\theta=0}^{1}(1-\theta)\beta_\xi''\Big(\partial_{x_i} u_\epsilon(s,x)+\theta\,\eta_\epsilon'(u_\epsilon(s,x);z)\partial_{x_i} u_\epsilon(s,x)\Big) \\
&\times\big(\eta_\epsilon'(u_\epsilon(s,x);z)\partial_{x_i} u_\epsilon(s,x)\big)^2\psi(x)\,d\theta\,\nu(dz)\,ds\,dx\Big] \\
&+ E\Big[\int_{\mathbb{R}^d_x}\int_{s=0}^{t}\partial_{x_i} u_\epsilon(s,x)\psi(x)\beta_\xi''\big(\partial_{x_i} u_\epsilon(s,x)\big)\nabla\partial_{x_i} u_\epsilon(s,x)\cdot F_\epsilon'(u_\epsilon(s,x))\,ds\,dx\Big] \\
&+ E\Big[\int_{\mathbb{R}^d_x}\int_{s=0}^{t}\partial_{x_i} u_\epsilon(s,x)\beta_\xi'\big(\partial_{x_i} u_\epsilon(s,x)\big)\nabla\psi(x)\cdot F_\epsilon'(u_\epsilon(s,x))\,ds\,dx\Big] \\
&+ \epsilon E\Big[\int_{\mathbb{R}^d_x}\int_{s=0}^{t}\beta_\xi\big(\partial_{x_i} u_\epsilon(s,x)\big)\Delta\psi(x)\,ds\,dx\Big] \\
:= E\Big[\int_{\mathbb{R}^d_x} \beta_\xi\big(\partial_{x_i} u_\epsilon(0,x)\big)\psi(x)\,dx\Big] &+ \mathcal{E}_1(\epsilon,\xi)+\mathcal{E}_2(\epsilon,\xi)+\mathcal{E}_3(\epsilon,\xi)+\mathcal{E}_4(\epsilon,\xi). \qquad (3.2)
\end{aligned}
$$

To estimate $\mathcal{E}_1(\epsilon,\xi)$, we proceed as follows. Note that we can rewrite $\mathcal{E}_1(\epsilon,\xi)$ as

$$
\mathcal{E}_1(\epsilon,\xi) = E\Big[\int_{\mathbb{R}^d_x}\int_{s=0}^{t}\int_{|z|>0}\int_{\theta=0}^{1}(1-\theta)\,h^2\beta_\xi''\big(a+\theta\,h\big)\psi(x)\,d\theta\,\nu(dz)\,ds\,dx\Big],
$$

where $a=\partial_{x_i} u_\epsilon(s,x)$ and $h=\eta_\epsilon'(u_\epsilon(s,x);z)\partial_{x_i} u_\epsilon(s,x)$. In view of the assumption (**A**.3), it is easy to see that

$$
h^2\beta_\xi''(a+\theta\,h) \le \big|\partial_{x_i} u_\epsilon(s,x)\big|^2(1\wedge|z|^2)\beta_\xi''(a+\theta\,h). \tag{3.3}
$$

Next we move on to find a suitable upper bound on $a^2\beta_\xi''\big(a+\theta\,h\big)$. Since β'' is an even function, without loss of generality we may assume that $a>0$. Then by our assumption (**A**.3)

$$
\partial_{x_i} u_\epsilon(t,x)+\theta\eta_\epsilon'\big(u_\epsilon(t,x);z\big)\partial_{x_i} u_\epsilon(t,x) \ge (1-\lambda^*)\partial_{x_i} u_\epsilon(t,x),
$$

for $\theta\in[0,1]$. In other words

$$
0 \le a \le (1-\lambda^*)^{-1}(a+\theta\,h). \tag{3.4}
$$

Combining (3.3) and (3.4) yields

$$h^2\beta_\xi''(a+\theta h) \le (1\wedge |z|^2)(1-\lambda^*)^{-2}(a+\theta h)^2\beta_\xi''(a+\theta h) \le C(1\wedge |z|^2)\,\xi.$$

Since by assumption (**A**.4), $\int_{|z|>0}(1\wedge |z|^2)\,\nu(dz) < +\infty$, we conclude that $\mathcal{E}_1(\epsilon,\xi) \mapsto 0$, as $\xi \downarrow 0$. Next, a similar argument (as described in Chen et al. [6, Theorem 1]) reveals that

$$\begin{aligned}
&\mathcal{E}_2(\epsilon,\xi) \mapsto 0, \text{ as } \xi \downarrow 0,\\
&|\mathcal{E}_3(\epsilon,\xi)| + |\mathcal{E}_4(\epsilon,\xi)| \le C(\epsilon,T)\int_{\mathbb{R}^d}\int_0^t \Big(|\nabla\psi(x)| + |\Delta\psi(x)|\Big)|\partial_{x_i}u_\epsilon(s,x)|\,dx\,ds.
\end{aligned}\tag{3.5}$$

Finally, keeping in mind (2.7), we combine the findings above to let $\xi \mapsto 0$ in (3.2) and conclude

$$\begin{aligned}
E\Big[\int_{\mathbb{R}^d_x}\big|\partial_{x_i}u_\epsilon(t,x)\big|\psi(x)\,dx\Big] &\le E\Big[\int_{\mathbb{R}^d_x}\big|\partial_{x_i}u_\epsilon(0,x)\big|\psi(x)\,dx\Big]\\
&\quad + C\int_{\mathbb{R}^d_x}\int_0^t \Big(|\nabla\psi(x)| + |\Delta\psi(x)|\Big)|\partial_{x_i}u_\epsilon(s,x)|\,dx\,ds
\end{aligned}\tag{3.6}$$

We replace ψ in (3.6) by the standard smooth cut-off function of $B_N(0)$ with support inside $B_{2N}(0)$ and let N go to ∞ and apply dominated convergence. The end result is our desired conclusion, i.e.

$$E\Big[\int_{\mathbb{R}^d_x}\big|\partial_{x_i}u_\epsilon(t,x)\big|\,dx\Big] \le E\Big[\int_{\mathbb{R}^d_x}\big|\partial_{x_i}u_\epsilon(0,x)\big|\,dx\Big]. \qquad \square$$

An important and immediate corollary of the uniform spatial BV estimate is the existence of BV bounds for the entropy solution of (1.1). We have the following theorem.

Theorem 3.2 *(BV entropy solution). Suppose that the assumptions* (**A**.2), (**A**.3)*, and* (**A**.4) *hold. Then there exists a unique entropy solution of* (1.1) *with initial data satisfying assumption* (**A**.1) *such that*

$$E\Big[|u(t,\cdot)|_{BV(\mathbb{R}^d)}\Big] \le E\Big[|u_0|_{BV(\mathbb{R}^d)}\Big], \text{ for any } t>0. \tag{3.7}$$

Proof. We take advantage of the well-posedness results from [2] and claim that the sequence $\{u_\epsilon(t,\cdot)\}$ converges, in the sense of Young measures, to the unique $L^p(\mathbb{R}^d)$-valued entropy solution $u(t,\cdot)$. In view of the uniform BV estimate in Theorem 3.1, by passing to the limit, we conclude (3.7). $\square$

4. Proof of the Main Theorem

The average L^1-contraction principle (cf. [2]) could be viewed as continuous dependence on the initial data for problems of type (1.1). A continuous dependence result involving the flux function and the noise coefficient is a subtle one and the proof requires the regularized problem:

$$\begin{cases} dv_\epsilon(s,y) + \operatorname{div}_y G_\epsilon(v_\epsilon(s,y))\,ds = \int_{|z|>0} \sigma_\epsilon(v_\epsilon(s,y);z)\tilde{N}(dz,ds) + \epsilon\Delta_{yy} v_\epsilon(s,y)\,ds, \\ \quad (s,y) \in \Pi_T, \\ v_\epsilon(0,y) = v_0^\epsilon(y), \quad y \in \mathbb{R}^d; \end{cases} \tag{4.1}$$

where $(v_0^\epsilon, \sigma_\epsilon, G_\epsilon)$ are regularized version of (v_0, σ, G) satisfying the conditions in (2.2). In view of Theorem 3.2, we conclude that $v_\epsilon(s,y)$ converges, as Young measures, to the unique BV-entropy solution $v(s,y)$ of (2.4) with initial data $v_0(y)$. Let $u(t,\cdot)$ be the unique BV-entropy solution of (1.1) with initial data $u_0(x)$. Moreover, assume that the assumptions (**A**.1)–(**A**.4) hold for both sets of given functions (v_0, G, σ) and (u_0, F, η).

We estimate the L^1-norm of $u - v$ and the proof is done by adapting the method of "*doubling of variables*" to the stochastic case as laid out in [6]. Likewise in [2], one needs to directly compare one entropy solution to the viscous approximation of the other. This approach is somewhat different from the deterministic approach (cf. [4,7,5,13]), where one can directly compare two entropy solutions.

To begin with, let ρ and ϱ be the standard mollifiers on $\mathbb{R}$ and $\mathbb{R}^d$ respectively such that $\operatorname{supp}(\rho) \subset [-1,0)$ and $\operatorname{supp}(\varrho) = B_1(0)$. For $\delta > 0$ and $\delta_0 > 0$, let $\rho_{\delta_0}(r) = \frac{1}{\delta_0}\rho(\frac{r}{\delta_0})$ and $\varrho_\delta(x) = \frac{1}{\delta^d}\varrho(\frac{x}{\delta})$. For a nonnegative test function $\psi \in C_c^{1,2}([0,\infty)\times\mathbb{R}^d)$ with $|\nabla\psi(t,x)| \le C\,\psi(t,x)$, $|\Delta\psi(t,x)| \le C\,\psi(t,x)$ and two positive constants δ, δ_0, define

$$\phi_{\delta,\delta_0}(t,x,s,y) = \rho_{\delta_0}(t-s)\varrho_\delta(x-y)\psi(s,y). \tag{4.2}$$

Observe that $\rho_{\delta_0}(t-s) \neq 0$ only if $s - \delta_0 \le t \le s$, and therefore $\phi_{\delta,\delta_0}(t,x;s,y) = 0$ outside $s - \delta_0 \le t < s$.

Furthermore, let ς be the standard symmetric nonnegative mollifier on $\mathbb{R}$ with support in $[-1,1]$ and $\varsigma_l(r) = \frac{1}{l}\varsigma(\frac{r}{l})$ for $l > 0$. We now write the entropy inequality for $u(t,x)$, based on the entropy pair $(\beta(\cdot - k), F^\beta(\cdot,k))$, then multiply by $\varsigma_l(v_\epsilon(s,y)-k)$ and integrate. The result is

$$\begin{aligned} 0 \le\ & E\Big[\int_{\Pi_T}\int_{\mathbb{R}_x^d}\int_{\mathbb{R}_k} \beta(u(0,x)-k)\phi_{\delta,\delta_0}(0,x,s,y)\varsigma_l(v_\epsilon(s,y)-k)\,dk\,dx\,dy\,ds\Big] \\ & + E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k} \beta(u(t,x)-k)\partial_t\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(v_\epsilon(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big] \\ & + E\Big[\int_{\Pi_T}\int_{\mathbb{R}_k}\int_{\Pi_T}\int_{|z|>0} \Big(\beta\big(u(t,x)+\eta(u(t,x);z)-k\big) - \beta(u(t,x)-k)\Big) \\ & \times \phi_{\delta,\delta_0}(t,x,s,y)\,\varsigma_l(v_\epsilon(s,y)-k)\,\tilde{N}(dz,dt)\,dx\,dk\,dy\,ds\Big] \end{aligned}$$

$$
\begin{aligned}
&+ E\Big[\int_{\Pi_T}\int_{t=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\Big(\beta\big(u(t,x)+\eta(u(t,x);z)-k\big)-\beta(u(t,x)-k)\\
&-\eta(u(t,x);z)\beta'(u(t,x)-k)\Big)\phi_{\delta,\delta_0}(t,x;s,y)\\
&\times \varsigma_l(v_\epsilon(s,y)-k)\,dk\,dx\,\nu(dz)\,dt\,dy\,ds\Big]\\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k} F^\beta(u(t,x),k)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\,\rho_{\delta_0}(t-s)\\
&\times \varsigma_l(v_\epsilon(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
=:\ & I_1+I_2+I_3+I_4+I_5.
\end{aligned}
\tag{4.3}
$$

We now apply the Itô–Lévy formula to (4.1) and multiply with test function $\phi_{\delta_0,\delta}$ and $\varsigma_l(u(t,x)-k)$ and integrate. The result is

$$
\begin{aligned}
0 \le\ & E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\beta(v_\epsilon(0,y)-k)\phi_{\delta,\delta_0}(t,x,0,y)\varsigma_l(u(t,x)-k)\,dk\,dx\,dy\,dt\Big]\\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k}\beta(v_\epsilon(s,y)-k)\partial_s\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(u(t,x)-k)\,dk\,dy\,ds\,dx\,dt\Big]\\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{|z|>0}\int_{\mathbb{R}_k}\Big(\beta\big(v_\epsilon(s,y)+\sigma_\epsilon(v_\epsilon(s,y);z)-k\big)-\beta(v_\epsilon(s,y)-k)\Big)\\
&\times\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(u(t,x)-k)\,dk\,\tilde N(dz,ds)\,dy\,dx\,dt\Big]\\
&+ E\Big[\int_{\Pi_T}\int_{s=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_k}\Big(\beta\big(v_\epsilon(s,y)+\sigma_\epsilon(v_\epsilon(s,y);z)-k\big)-\beta(v_\epsilon(s,y)-k)\\
&-\sigma_\epsilon(v_\epsilon(s,y);z)\beta'(v_\epsilon(s,y)-k)\Big)\phi_{\delta,\delta_0}(t,x;s,y)\\
&\times\varsigma_l(u(t,x)-k)\,dk\,dy\,\nu(dz)\,ds\,dx\,dt\Big]\\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k} G^\beta_\epsilon(v_\epsilon(s,y),k)\cdot\nabla_y\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s)\\
&\times\varsigma_l(u(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k} G^\beta_\epsilon(v_\epsilon(s,y),k)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\rho_{\delta_0}(t-s)\\
&\times\varsigma_l(u(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]
\end{aligned}
$$

$$- \epsilon E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k} \beta'(v_\epsilon(s,y)-k)\nabla_y v_\epsilon(s,y)\cdot\nabla_y\phi_{\delta,\delta_0}(t,x,s,y)$$

$$\times\, \varsigma_l(u(t,x)-k)\,dk\,dy\,ds\,dx\,dt\Big], \tag{4.4}$$

where $G_\epsilon^\beta(a,b)=\int_a^b \beta'(r-b)G_\epsilon'(r)\,dr$. It follows by direct computations that there is $p\in\mathbb{N}$ such that

$$\big|G_\epsilon^\beta(a,b)-G^\beta(a,b)\big|\le C\epsilon\big(1+|a|^{2p}+|b|^{2p}\big).$$

In view of the uniform moment estimates, it follows from (4.4) that

$$0 \le E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k} \beta(v_\epsilon(0,y)-k)\phi_{\delta,\delta_0}(t,x,0,y)\varsigma_l(u(t,x)-k)\,dk\,dx\,dy\,dt\Big]$$

$$+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k} \beta(v_\epsilon(s,y)-k)\partial_s\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(u(t,x)-k)\,dk\,dy\,ds\,dx\,dt\Big]$$

$$+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{|z|>0}\int_{\mathbb{R}_k}\Big(\beta\big(v_\epsilon(s,y)+\sigma_\epsilon(v_\epsilon(s,y);z)-k\big)-\beta(v_\epsilon(s,y)-k)\Big)$$

$$\times\,\phi_{\delta,\delta_0}(t,x,s,y)\varsigma_l(u(t,x)-k)\,dk\,\tilde N(dz,ds)\,dy\,dx\,dt\Big]$$

$$+ E\Big[\int_{\Pi_T}\int_{s=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_k}\Big(\beta\big(v_\epsilon(s,y)+\sigma_\epsilon(v_\epsilon(s,y);z)-k\big)-\beta(v_\epsilon(s,y)-k)$$

$$-\,\sigma_\epsilon(v_\epsilon(s,y);z)\beta'(v_\epsilon(s,y)-k)\Big)\phi_{\delta,\delta_0}(t,x;s,y)\varsigma_l(u(t,x)-k)\,dk\,dy\,\nu(dz)\,ds\,dx\,dt\Big]$$

$$+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k} G^\beta(v_\epsilon(s,y),k)\cdot\nabla_y\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s)$$

$$\times\,\varsigma_l(u(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]$$

$$+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k} G^\beta(v_\epsilon(s,y),k)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\rho_{\delta_0}(t-s)$$

$$\times\,\varsigma_l(u(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]$$

$$-\,\epsilon E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k}\beta'(v_\epsilon(s,y)-k)\nabla_y v_\epsilon(s,y)\cdot\nabla_y\phi_{\delta,\delta_0}\varsigma_l(u(t,x)-k)\,dk\,dy\,ds\,dx\,dt\Big]$$

$$+\,C(\beta,\psi)\frac{\epsilon}{\delta}$$

$$=: J_1+J_2+J_3+J_4+J_5+J_6+J_7+C(\beta,\psi)\frac{\epsilon}{\delta}, \tag{4.5}$$

where $C(\beta,\psi)$ is a constant depending only on β and ψ. Our aim is to add (4.3) and (4.5), and pass to the limits with respect to the various parameters involved. We do this by claiming a series of lemmas and proofs of these assertions follow from [2, Section 5] modulo cosmetic changes.

Lemma 4.1. *It holds that $J_1 = 0$ and*

$$\lim_{l\downarrow 0}\lim_{\delta_0\downarrow 0}\Big(I_1+J_1\Big) = E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\beta(u(0,x)-v_\epsilon(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big],$$

$$\lim_{l\downarrow 0}\lim_{\delta_0\downarrow 0}\Big(I_2+J_2\Big) = E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\beta(v_\epsilon(s,y)-u(s,x))\partial_s\psi(s,y)\,\varrho_\delta(x-y)\,dy\,dx\,ds\Big].$$

Lemma 4.2. *The following hold:*

$$\lim_{l\to 0}\lim_{\delta_0\to 0} I_5 = E\Big[\int_{s=0}^{T}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x} F^\beta(u(s,x),v_\epsilon(s,y))\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\,dx\,dy\,ds\Big], \tag{4.6}$$

$$\lim_{l\to 0}\lim_{\delta_0\to 0} J_5 = E\Big[\int_{s=0}^{T}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x} G^\beta(v_\epsilon(s,y),u(s,x))\cdot\nabla_y\varrho_\delta(x-y)\,\psi(s,y)\,dx\,dy\,ds\Big], \tag{4.7}$$

$$\lim_{l\to 0}\lim_{\delta_0\to 0} J_6 = E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x} G^\beta(v_\epsilon(s,y),u(s,x))\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\,dx\,dy\,ds\Big].$$

Thanks to the uniform spatial BV estimate in Theorem 3.1, we conclude that

$$\begin{aligned}|J_7| &\le \epsilon||\beta'||_\infty\Big|E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}|\nabla_y v_\epsilon(s,y)||\nabla_y[\psi(s,y)\varrho_\delta(x-y)|\,dx\,dy\,ds\Big]\Big|\\ &\le C\frac{\epsilon}{\delta}E\big[|v_0|_{BV(\mathbb{R}^d)}\big].\end{aligned} \tag{4.8}$$

Lemma 4.3. *It holds that*

$$\begin{aligned}\lim_{l\to 0}\lim_{\delta_0\to 0} J_4 = E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\lambda=0}^{1}&(1-\lambda)\beta''\big(v_\epsilon(s,y)-u(s,x)+\lambda\sigma_\epsilon(v_\epsilon(s,y);z)\big)\\ &\times|\sigma_\epsilon(v_\epsilon(s,y);z)|^2\psi(s,y)\varrho_\delta(x-y)\,d\lambda\,\nu(dz)\,dx\,dy\,ds\Big],\end{aligned} \tag{4.9}$$

$$\lim_{l\to 0}\lim_{\delta_0\to 0} I_4 = E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\lambda=0}^{1}(1-\lambda)\beta''\big(u(s,x)-v_\epsilon(s,y)+\lambda\eta(u(s,x);z)\big)$$
$$\times |\eta(u(s,x);z)|^2\psi(s,y)\varrho_\delta(x-y)\,d\lambda\,\nu(dz)\,dx\,dy\,ds\Big]. \tag{4.10}$$

Finally, we consider the stochastic term $I_3 + J_3$;

Lemma 4.4. *It holds that* $J_3 = 0$ *and*

$$\lim_{l\to 0}\lim_{\delta_0\to 0} I_3 = E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\Big(\beta(u(r,x)+\eta(u(r,x);z)-v_\epsilon(r,y)-\sigma_\epsilon(v_\epsilon(r,y);z))$$
$$-\beta(u(r,x)-v_\epsilon(r,y)-\sigma_\epsilon(v_\epsilon(r,y);z))+\beta\big(u(r,x)-v_\epsilon(r,y)\big)$$
$$-\beta\big(u(r,x)+\eta(u(r,x);z)-v_\epsilon(r,y)\big)\Big)\psi(r,y)\,\varrho_\delta(x-y)\,\nu(dz)\,dx\,dy\,dr\Big].$$

To proceed further, we combine Lemma 4.4 and Lemma 4.3 and conclude that

$$\lim_{l\to 0}\lim_{\delta_0\to 0}\Big((I_3+J_3)+(I_4+J_4)\Big)$$
$$= E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\Big(\int_{|z|>0}\Big\{\beta\big(u(t,x)-v_\epsilon(t,y)+\eta(u(t,x);z)-\sigma_\epsilon(v_\epsilon(t,y);z)\big)$$
$$-\beta\big(u(t,x)-v_\epsilon(t,y)\big)-\big(\eta(u(t,x);z)-\sigma_\epsilon(v_\epsilon(t,y);z)\big)$$
$$\times\beta'\big(u(t,x)-v_\epsilon(t,y)\big)\Big\}\,\nu(dz)\Big)\psi(t,y)\varrho_\delta(x-y)\,dx\,dy\,dt\Big]$$
$$= E\Big[\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\rho=0}^{1}\beta''\Big(u(r,x)-v_\epsilon(r,y)+\rho\big(\eta(u(r,x);z)-\sigma_\epsilon(v_\epsilon(r,y);z)\big)\Big)$$
$$\times(1-\rho)\big|\eta(u(r,x);z)-\sigma_\epsilon(v_\epsilon(r,y);z)\big|^2\psi(r,y)\,\varrho_\delta(x-y)\,d\rho\,dx\,dy\,\nu(dz)\,dr\Big]. \tag{4.11}$$

We are now in a position to add (4.3) and (4.5) and pass to the limits $\lim\limits_{l\downarrow 0}\lim\limits_{\delta_0\downarrow 0}$. In what follows, invoking Lemma 4.1, Lemma 4.2 and the expressions (4.8) and (4.11), we arrive at

$$0 \le E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\beta(u(0,x)-v_\epsilon(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big]$$
$$+E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\beta(v_\epsilon(s,y)-u(s,x))\partial_s\psi(s,y)\varrho_\delta(x-y)\,dy\,dx\,ds\Big]$$

$$
\begin{aligned}
& - E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_y} \nabla_y \cdot \big\{ G^\beta\big(v_\epsilon(s,y), u(s,x)\big) - F^\beta\big(u(s,x), v_\epsilon(s,y)\big)\big\} \\
& \times \psi(s,y) \varrho_\delta(x-y)\,dy\,dx\,ds \Big] \\
& + E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_y} F^\beta\big(u(s,x), v_\epsilon(s,y)\big) \cdot \nabla_y \psi(s,y)\, \varrho_\delta(x-y)\,dy\,dx\,ds \Big] \\
& + C\Big(E\big[|v_0|_{BV(\mathbb{R}^d)}\big] + 1\Big) \frac{\epsilon}{\delta} \\
& + E\Big[\int_{r=0}^{T} \int_{|z|>0} \int_{\mathbb{R}^d_y} \int_{\mathbb{R}^d_x} \int_{\rho=0}^{1} \beta''\Big(u(r,x) - v_\epsilon(r,y) + \rho\big(\eta(u(r,x);z) - \sigma_\epsilon(v_\epsilon(r,y);z)\big)\Big) \\
& \times (1-\rho)\big|\eta(u(r,x);z) - \sigma_\epsilon(v_\epsilon(r,y);z)\big|^2 \psi(r,y)\, \varrho_\delta(x-y)\,d\rho\,dx\,dy\,\nu(dz)\,dr \Big] \\
:= {} & \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5 + C\Big(E\big[|v_0|_{BV(\mathbb{R}^d)}\big] + 1\Big) \frac{\epsilon}{\delta}. \qquad (4.12)
\end{aligned}
$$

Again, our aim is to estimate all the above terms suitably. First observe that, since $\beta_\xi(r) \le |r|$, we obtain

$$
|\mathcal{A}_1| \le E\Big[\int_{\mathbb{R}^d_y} \int_{\mathbb{R}^d_x} \big|v_\epsilon(0,y) - u(0,x)\big| \psi(0,y)\, \varrho_\delta(x-y)\,dx\,dy \Big]. \qquad (4.13)
$$

Next, following computations as in Chen et al. [6, Section 5], it can be shown that

$$
|\mathcal{A}_3| \le E\Big[|v_0|_{BV(\mathbb{R}^d)}\Big] \Big(M_2\, \xi\, ||F''||_\infty + ||F' - G'||_\infty \Big) \int_{s=0}^{T} ||\psi(s,\cdot)||_{L^\infty(\mathbb{R}^d)}\,ds. \qquad (4.14)
$$

Note that $|\nabla \psi(t,x)| \le C\, \psi(t,x)$ and $|F^\beta(a,b)| \le ||F'||_\infty |a-b|$ for any $a, b \in \mathbb{R}$. Therefore, we conclude

$$
\begin{aligned}
|\mathcal{A}_4| \le {} & C||F'||_{L^\infty} E\Big[\int_0^T \int_{\mathbb{R}^d_x \times \mathbb{R}^d_y} \beta_\xi\big(u(s,x) - v_\epsilon(s,y)\big) \psi(s,y) \varrho_\delta(x-y)\,dx\,dy\,ds \Big] \\
& + C M_1\, ||F'||_{L^\infty}\, \xi \int_{s=0}^{T} ||\psi(s,\cdot)||_{L^\infty(\mathbb{R}^d)}\,ds. \qquad (4.15)
\end{aligned}
$$

We now define $a := u(r,x) - v_\epsilon(r,y)$ and $b := \eta(u(r,x);z) - \sigma_\epsilon(v_\epsilon(r,y);z)$, and observe

$$\begin{aligned}
\mathcal{A}_5 &= E\Big[\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\rho=0}^{1}(1-\rho)b^2\beta''\big(a+\rho\, b\big)\psi(r,y)\,\varrho_\delta(x-y)\,d\rho\,dx\,dy\,\nu(dz)\,dr\Big]\\
&\le C E\Big[\int_{0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\rho=0}^{1}\big|\eta(u(r,x);z)-\sigma(u(r,x);z)\big|^2\beta''\big(a+\rho\, b\big)\\
&\quad \times \psi(r,y)\,\varrho_\delta(x-y)\,d\rho\,dx\,dy\,\nu(dz)\,dr\Big]\\
&\quad + C E\Big[\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\rho=0}^{1}\big|\sigma(u(r,x);z)-\sigma(v_\epsilon(r,y);z)\big|^2\beta''\big(a+\rho\, b\big)\\
&\quad \times \psi(r,y)\,\varrho_\delta(x-y)\,d\rho\,dx\,dy\,\nu(dz)\,dr\Big]\\
&\quad + C E\Big[\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\rho=0}^{1}\big|\sigma(v_\epsilon(r,y);z)-\sigma_\epsilon(v_\epsilon(r,y);z)\big|^2\beta''\big(a+\rho\, b\big)\\
&\quad \times \psi(r,y)\,\varrho_\delta(x-y)\,d\rho\,dx\,dy\,\nu(dz)\,dr\Big]\\
&:= \mathcal{A}_5^1 + \mathcal{A}_5^2 + \mathcal{A}_5^3. \qquad (4.16)
\end{aligned}$$

We now recall that $\mathcal{D}(\eta,\sigma) = \sup_{u\in\mathbb{R}}\int_{|z|>0}\frac{|\eta(u,z)-\sigma(u,z)|^2}{1+|u|^2}\nu(dz)$. It is easy to see that

$$\begin{aligned}
\mathcal{A}_5^1 &\le \frac{C\mathcal{D}(\eta,\sigma)}{\xi}E\Big[\int_{r=0}^{T}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_y}(1+|u(r,x)|^2)\psi(r,y)\rho_\delta(x-y)\,dy\,dx\,dr\Big]\\
&\le \frac{C\mathcal{D}(\eta,\sigma)}{\xi}\Big(\int_0^T||\psi(s,\cdot)||_{L^1}\,ds + \int_0^T||\psi(r,\cdot)||_\infty\,dr\Big). \qquad (4.17)
\end{aligned}$$

Next, we move on to estimate the term $\mathcal{A}_5^2$. Observe that, by (**A**.3),

$$\big|\sigma(u(r,x);z)-\sigma(v_\epsilon(r,y);z)\big|^2\beta''(a+\rho\, b) \le (1\wedge|z|^2)\,a^2\,\beta''(a+\rho\, b). \qquad (4.18)$$

We want to find an upper bound on $a^2\,\beta''(a+\rho\, b)$. As β'' is non-negative and symmetric around zero, without loss of generality, we may assume that $a>0$. Then, by our assumption (**A**.3), we conclude that

$$\big|\eta(u(r,x);z)-\sigma_\epsilon(v_\epsilon(r,y);z)\big|\le\big|\eta(u(r,x);z)-\sigma(u(r,x);z)\big|+\lambda^* a+C\epsilon(1+|v_\epsilon|),$$

which implies that $a+\rho b\ge-\big|\eta(u(r,x);z)-\sigma(u(r,x);z)\big|-C\epsilon(1+|v_\epsilon|)+(1-\lambda^*)a$ for $\rho\in[0,1]$. In other words

$$0\le a\le(1-\lambda^*)^{-1}\Big\{a+\rho b+\big|\eta(u(r,x);z)-\sigma(u(r,x);z)\big|+C\epsilon(1+|v_\epsilon|)\Big\}. \tag{4.19}$$

Now, we shall make use of (4.19) in (4.18), to obtain

$$\begin{aligned}&\big|\sigma(u(r,x);z)-\sigma(v_\epsilon(r,y);z)\big|^2\beta_\xi''(a+\rho b)\\ &\le C\Big(\xi+\frac{\big|\eta(u(r,x);z)-\sigma(u(r,x);z)\big|^2}{\xi}+\frac{\epsilon^2\left(1+|v_\epsilon|^2\right)}{\xi}\Big)(1\wedge|z|^2).\end{aligned}$$

This helps us to conclude

$$\begin{aligned}\big|\mathcal{A}_5^2\big|&\le CE\Big[\int_r\int_{|z|>0}\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d}\Big(\xi+\frac{\epsilon^2\left(1+|v_\epsilon|^2\right)}{\xi}\Big)(1\wedge|z|^2)\psi(r,y)\,\varrho_\delta(x-y)\,dx\,dy\,m(dz)\,dr\Big]\\ &\quad+\frac{\mathcal{D}(\eta,\sigma)}{\xi}\int_0^T\int_{\mathbb{R}_x^d}\int_{\mathbb{R}_y^d}(1+|u(r,x)|^2)\psi(r,y)\rho_\delta(x-y)\,dx\,dy\,dr\\ &\le C(\xi+\frac{\epsilon^2}{\xi})\int_{s=0}^T||\psi(s,\cdot)||_{L^\infty(\mathbb{R}^d)}\,ds+\frac{C\mathcal{D}(\eta,\sigma)}{\xi}\Big(\int_0^T||\psi(s,\cdot)||_{L^1}\,ds+\int_0^T||\psi(r,\cdot)||_\infty\,dr\Big).\end{aligned} \tag{4.20}$$

Next, we move on to estimate the term $\mathcal{A}_5^3$. In fact, it follows easily that

$$|\mathcal{A}_5^3|\le C\frac{\epsilon^2}{\xi}\int_0^T||\psi(s,\cdot)||_{L^\infty}\,ds. \tag{4.21}$$

We now make use of the estimates (4.17), (4.20) and (4.21). Then it is evident from (4.16) that

$$\begin{aligned}|\mathcal{A}_5|&\le\frac{C\mathcal{D}(\eta,\sigma)}{\xi}\Big(\int_0^T||\psi(s,\cdot)||_{L^1}\,ds+\int_0^T||\psi(s,\cdot)||_\infty\,dr\Big)\\ &\quad+C(\xi+\frac{\epsilon^2}{\xi})\int_{s=0}^T||\psi(s,\cdot)||_{L^\infty(\mathbb{R}^d)}\,ds.\end{aligned} \tag{4.22}$$

We now use (4.13)–(4.15) and (4.22) in (4.12) and apply $\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0}$ to conclude

$$
\begin{aligned}
0 \le\ & E\Big[\int_{\mathbb{R}^d_x} \big|v_0(x) - u(0,x)\big|\psi(0,x)\,dx\Big] \\
&+ E\Big[|u_0|_{BV(\mathbb{R}^d)}\Big]\Big(M_2\,\xi\,||F''||_\infty + ||F'-G'||_\infty\Big)\int_{s=0}^{T} ||\psi(s,\cdot)||_{L^\infty(\mathbb{R}^d)}\,ds \\
&+ C||F'||_{L^\infty} E\Big[\int_{s=0}^{T}\int_{\mathbb{R}^d_x} \beta_\xi\big(v(s,x) - u(s,x)\big)\psi(s,x)\,dx\,ds\Big] \\
&+ C\big(M_1\,||F'||_{L^\infty} + 1\big)\,\xi \int_{s=0}^{T} ||\psi(s,\cdot)||_{L^\infty(\mathbb{R}^d)}\,ds \\
&+ \frac{C\mathcal{D}(\eta,\sigma)}{\xi}\Big(\int_0^T ||\psi(s,\cdot)||_{L^1}\,ds + \int_0^T ||\psi(r,\cdot)||_\infty\,dr\Big) \\
&+ E\Big[\int_{\Pi_T} \beta_\xi(u(s,x) - v(s,x))\partial_s\psi(s,x)\,dx\,ds\Big].
\end{aligned}
\tag{4.23}
$$

For a given function $\phi \in C_c^2(\mathbb{R}^d)$ satisfying $|\nabla\phi(x)| \le C\phi(x)$, $|\Delta\phi(x)| \le C\phi(x)$, we choose $\psi(t,x) = h(t)\phi(x)$ as per [2, Proof of Theorem 2.2] and apply a weaker version of Grownwall's inequality to obtain

$$
\begin{aligned}
& E\Big[\int_{\mathbb{R}^d_x} \beta_\xi\big(u(t,x) - v(t,x)\big)\phi(x)\,dx\Big] \\
&\quad \le e^{C\,t\,||F'||_\infty} E\Big[\int_{\mathbb{R}^d_x} \big|v_0(x) - u(0,x)\big|\phi(x)\,dx\Big] + M_1\,\xi\,||\phi(\cdot)||_{L^1(\mathbb{R}^d)} \\
&\qquad + Ce^{C||F'||_\infty t}\Big\{\big(M_1\,||F'||_{L^\infty} + 1\big)\,\xi\,||\phi(\cdot)||_{L^\infty(\mathbb{R}^d)}\,t \\
&\qquad + E\Big[|v_0|_{BV(\mathbb{R}^d)}\Big]\Big(M_2\,\xi\,||F''||_\infty + ||F'-G'||_\infty\Big)||\phi(\cdot)||_{L^\infty(\mathbb{R}^d)}\,t \\
&\qquad + \frac{C\,t\mathcal{D}(\eta,\sigma)}{\xi}\Big(||\phi||_{L^1} + ||\phi||_{L^\infty}\Big)\Big\},
\end{aligned}
\tag{4.24}
$$

for almost every $t > 0$. We now simply choose $\xi = \sqrt{t\mathcal{D}(\eta,\sigma)}$ in (4.24) to conclude (2.5) for a.e. $t > 0$ and thereby completing the proof.

5. Proof of the Main Corollary

It is already known that the vanishing viscosity solutions converge (in an appropriate sense) to the unique entropy solution of the stochastic conservation law. However, the nature of such convergence described by a rate of convergence is not available. As a by product of the Main Theorem, we explicitly obtain the rate of convergence of vanishing viscosity solutions to the unique BV-entropy solution of the underlying problem (1.1).

By similar arguments as in the proof of the Main Theorem (cf. Section 4), we arrive at

$$
\begin{aligned}
&E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x} \big|u_\epsilon(t,y)-u(t,x)\big|\phi(y)\varrho_\delta(x-y)\,dx\,dy\Big] \\
&\quad\le e^{C||F'||_{L^\infty} t}\Big\{E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x} \big|u_\epsilon(0,y)-u_0(x)\big|\phi(y)\,\varrho_\delta(x-y)\,dx\,dy\Big] + C\Big(1+E\big[|u_0|_{BV(\mathbb{R}^d)}\big]\Big)\frac{\epsilon}{\delta}\Big\} \\
&\qquad + Ce^{C||F'||_{L^\infty} t}\Big[\Big(1+E\big[|u_0|_{BV(\mathbb{R}^d)}\big]\Big)\xi\,||\phi(\cdot)||_{L^\infty(\mathbb{R}^d)}\,t + \frac{\epsilon^2}{\xi}||\phi(\cdot)||_{L^\infty(\mathbb{R}^d)}\,t\Big] \\
&\qquad + C\xi\,||\phi(\cdot)||_{L^\infty(\mathbb{R}^d)}.
\end{aligned} \tag{5.1}
$$

First sending $\phi \mapsto \chi_{\mathbb{R}^d}$, and then choosing $\xi = \epsilon$ in (5.1) yields

$$
\begin{aligned}
E\Big[\int_{\mathbb{R}^d_y} \big|u_\epsilon(t,y)-u(t,y)\big|\,dy\Big] \le e^{C||F'||_{L^\infty} t}&\Big\{E\Big[\int_{\mathbb{R}^d_y}\big|u_\epsilon(0,y)-u_0(y)\big|dy\Big] \\
&+ C\Big(1+E\big[|u_0|_{BV(\mathbb{R}^d)}\big]\Big)\frac{\epsilon}{\delta} + \delta\,E\big[|u_0|_{BV(\mathbb{R}^d)}\big]\Big\} \\
&+ Ce^{C||F'||_{L^\infty} t}\Big(1+E\big[|u_0|_{BV(\mathbb{R}^d)}\big]\Big)\epsilon\,t \\
&+ C\epsilon + \delta\,E\big[|u_0|_{BV(\mathbb{R}^d)}\big].
\end{aligned} \tag{5.2}
$$

We choose $\delta = \epsilon^{\frac{1}{2}}$ in (5.2), and conclude that, for a.e. $t > 0$,

$$
\begin{aligned}
E\Big[\int_{\mathbb{R}^d_x} \big|u_\epsilon(t,x)-u(t,x)\big|\,dx\Big] \le C(T)&\Big\{\epsilon^{\frac{1}{2}}\big(1+E[|u_0|_{BV(\mathbb{R}^d)}]\big)(1+t) \\
&+ E\Big[\int_{\mathbb{R}^d_x}\big|u_\epsilon(0,x)-u_0(x)\big|\,dx\Big]\Big\},
\end{aligned}
$$

for some constant $C(T) > 0$, independent of $E\big[|u_0|_{BV(\mathbb{R}^d)}\big]$. This completes the proof.

6. Fractional BV estimates

In this section, we focus on a more general form of (1.1), namely the problem

$$\begin{cases} du(t,x) + \operatorname{div}_x F(u(t,x))\,dt = \int_{|z|>0} \eta(x,u(t,x);z)\,\tilde{N}(dz,dt), & x \in \Pi_T, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{6.1}$$

Notably, the jump amplitude η depends explicitly on the spatial position x and the following compatibility conditions are needed.

(**B**.1) There exist positive constants $K > 0$ and $\lambda^* \in [0,1)$ such that

$$|\eta(x,u;z) - \eta(y,v;z)| \le (\lambda^*|u-v| + K|x-y|)(|z| \wedge 1),$$
$$\text{for all } u, v \in \mathbb{R}; \quad z \in \mathbb{R}; \quad x, y \in \mathbb{R}^d.$$

(**B**.2) There exists a non-negative function $g(x) \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that

$$|\eta(x,u;z)| \le g(x)(1+|u|)(|z| \wedge 1), \text{ for all } (x,u,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}.$$

The uniform BV bound in Theorem 3.1 is no longer available for the problem (6.1) and, as a result, our method to have continuous dependence estimate for (6.1) does not apply. However, one can work along the path laid out in [6] and obtain a fractional BV estimate for (6.1) and it involves obtaining uniform fractional BV estimate for the viscous problem

$$du_\epsilon(t,x) + \operatorname{div}_x F_\epsilon(u_\epsilon(t,x))\,dt = \int\limits_{|z|>0} \eta_\epsilon(x,u_\epsilon(t,x);z)\tilde{N}(dz,dt) + \epsilon\Delta_{xx}u_\epsilon(t,x)\,dt, \tag{6.2}$$

where F_ϵ, η_ϵ satisfy (2.2). The conclusions of Proposition 2.1 is still valid for (6.2). We now establish a uniform fractional BV estimation of solutions of (6.2).

Theorem 6.1 *(Fractional BV estimate). Let the assumptions* (**A**.1), (**A**.2), (**B**.1), (**B**.2)*, and* (**A**.4) *hold. Let u_ϵ be a solution of* (6.2) *with the initial data $u_0(x)$ belongs to the Besov space $B^\mu_{1,\infty}(\mathbb{R}^d)$ for some $\mu \in (\frac{1}{2}, 1)$. Moreover, we assume that $F''_\epsilon \in L^\infty$. Then, for fixed $T > 0$ and $R > 0$, there exit constants $C(T,R)$ and $r \in (0,1)$, independent of ϵ, such that for any $0 < t < T$,*

$$\sup_{|y| \le \delta} E\Big[\int\limits_{x \in B_R} \big|u_\epsilon(t,x+y) - u_\epsilon(t,x)\big|\,dx\Big] \le C(T,R)\,\delta^r.$$

Proof. Let $\phi(x) \in C^2_c(\mathbb{R}^d)$ be a test function such that $|\nabla\phi(x)| \le C\phi(x)$ and $|\Delta\phi(x)| \le C\phi(x)$ for some constant $C > 0$. Moreover, let $(J_\delta)_\delta$ be a sequence of standard mollifiers in $\mathbb{R}^d$. Define $\psi_\delta(x,y) := J_\delta\left(\frac{x-y}{2}\right)\phi\left(\frac{x+y}{2}\right)$. Following Chen et al. [6, Theorem 7], we subtract two solutions $u_\epsilon(t,x)$, $u_\epsilon(t,y)$ of (6.2), and apply Itô–Lévy formula to the resulting equation to obtain

$$E\Big[\int\limits_{\mathbb{R}^d_y}\int\limits_{\mathbb{R}^d_x} \beta_\xi\big(u_\epsilon(t,x) - u_\epsilon(t,y)\big)\psi_\delta(x,y)\,dx\,dy\Big]$$
$$- E\Big[\int\limits_{\mathbb{R}^d_y}\int\limits_{\mathbb{R}^d_x} \beta_\xi\big(u_\epsilon(0,x) - u_\epsilon(0,y)\big)\psi_\delta(x,y)\,dx\,dy\Big]$$

$$\leq C\big((1+\xi)||F'||_\infty+\epsilon\big)\int_{s=0}^{t} E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\big|u_\epsilon(s,x)-u_\epsilon(s,y)\big|\phi(\frac{x+y}{2})J_\delta(\frac{x-y}{2})\,dx\,dy\Big]\,ds$$

$$+C||F'||_\infty\xi\,E\Big[\int_{s=0}^{t}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\big|u_\epsilon(s,x)-u_\epsilon(s,y)\big|\phi(\frac{x+y}{2})|\nabla_y J_\delta(\frac{x-y}{2})|\,dx\,dy\,ds\Big]$$

$$+E\Big[\int_{r=0}^{t}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\rho=0}^{1}\beta''_\xi\Big(u_\epsilon(r,x)-u_\epsilon(r,y)+\rho\big(\eta_\epsilon(x,u_\epsilon(r,x);z)-\eta_\epsilon(y,u_\epsilon(r,y);z)\big)\Big)$$

$$\times\big|\eta_\epsilon(x,u_\epsilon(r,x);z)-\eta_\epsilon(y,u_\epsilon(r,y);z)\big|^2\psi_\delta(x,y)\,d\rho\,dx\,dy\,\nu(dz)\,dr\Big]. \tag{6.3}$$

Next, we mainly focus on the last term of (6.3). Rest of the terms can be treated as in Chen et al. [6, Theorem 7]. In what follows, we first let $a=u_\epsilon(t,x)-u_\epsilon(t,y)$ and $b=\eta_\epsilon(x,u_\epsilon(t,x);z)-\eta_\epsilon(y,u_\epsilon(t,y);z)$. Observe that

$$\begin{aligned}
b^2\beta''_\xi(a+\rho\,b) &= (\eta_\epsilon(x,u_\epsilon(t,x);z)-\eta_\epsilon(y,u_\epsilon(t,y);z))^2\,\beta''_\xi(a+\rho\,b)\\
&\leq \Big(|u_\epsilon(t,x)-u_\epsilon(t,y)|^2+K^2|x-y|^2\Big)(1\wedge|z|^2)\,\beta''_\xi(a+\rho\,b)\\
&= \Big(a^2+K^2|x-y|^2\Big)\,\beta''_\xi(a+\rho\,b)\,(1\wedge|z|^2).
\end{aligned} \tag{6.4}$$

As before (cf. (3.4)), one can use assumption (**B**.1) on $\eta(x,u;z)$ to conclude

$$0\leq a\leq(1-\lambda^*)^{-1}\big(a+\rho b+K|x-y|\big).$$

In view of (6.4), we have

$$\begin{aligned}
b^2\beta''_\xi(a+\rho\,b) &\leq (1-\lambda^*)^{-2}\big(a+\rho\,b+K|x-y|\big)^2\,\beta''_\xi(a+\rho b)\,(|z|^2\wedge 1)\\
&\quad+\frac{K|x-y|^2}{\xi}\,(|z|^2\wedge 1)\\
&\leq\Big[2(1-\lambda^*)^{-2}C\xi+C(K,\lambda^*)\frac{|x-y|^2}{\xi}\Big](|z|^2\wedge 1),
\end{aligned}$$

and hence

$$E\Big[\int_{r=0}^{t}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\rho=0}^{1}b^2\beta''_\xi(a+\rho\,b)\psi_\delta(x,y)\,d\rho\,dx\,dy\,\nu(dz)\,dr\Big]$$

$$\leq E\Big[\int_{r=0}^{t}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\Big\{2(1-\lambda^*)^{-2}C\xi+C(K,\lambda^*)\frac{|x-y|^2}{\xi}\Big\}$$

$$\times (|z|^2 \wedge 1)\psi_\delta(x,y)\,dx\,dy\,\nu(dz)\,dr\Big]$$
$$\le C_1\Big(\xi + \frac{\delta^2}{\xi}\Big)t\,||\phi(\cdot)||_{L^\infty(\mathbb{R}^d)}. \tag{6.5}$$

Finally, keeping in mind the estimate (6.5), a simple application of the lemma [6, Lemma 2] gives

$$\sup_{|y|\le\delta} E\Big[\int_{\mathbb{R}^d_x} \big|u_\epsilon(t,x+y) - u_\epsilon(t,x)\big|\phi(x)\,dx\Big]$$
$$\le C(T)\,\delta^r\bigg[\Big(E\Big[||u_0||_{B^\mu_{1,\infty}(\mathbb{R}^d)}\Big]+1\Big)||\phi||_{L^\infty(\mathbb{R}^d)} + ||\phi||_{L^1(\mathbb{R}^d)}\bigg]$$
$$+ C_2\,\delta^r\,E\Big[||u_\epsilon(t,\cdot)||_{L^1(\mathbb{R}^d)}\Big].$$

To proceed further, let $K_R = \{x : |x| \le R\}$. Choose $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(x) = 1$ on K_R. Then, for $r < \frac{1}{2}$, we have

$$\sup_{|y|\le\delta} E\bigg[\int_{K_R} \big|u_\epsilon(t,x+y) - u_\epsilon(t,x)\big|\,dx\bigg] \le C(T,R)\,\delta^r,$$

which completes the proof. □

In view of the well-posedness results from [2], the L^p-valued entropy solution of (6.1) satisfies the fractional BV estimate in Theorem 6.1. In other words, we have the following theorem.

Theorem 6.2. *Suppose that the assumptions* (**A**.2), (**A**.3), (**A**.4), (**B**.1), *and* (**B**.2) *hold and the initial data u_0 belong to the Besov space $B^\mu_{1,\infty}(\mathbb{R}^d)$ for some $\mu \in (\frac{1}{2}, 1)$ and $E\Big[\|u_0\|^p_{L^p(\mathbb{R}^d)} + \|u_0\|^p_{L^2(\mathbb{R}^d)}\Big] < \infty$ for $p = 1, 2, \cdots$. Then the problem (6.1) admits an entropy solution $u(t,\cdot)$ such that*

$$\sup_{0\le t\le T} E\Big[\|u(t,\cdot)\|^p_{L^p(\mathbb{R}^d)}\Big] < \infty, \text{ for } p = 1, 2, \cdots.$$

Moreover, there exist constants C_T^R and $r \in (0, \frac{1}{2})$ such that, for almost every $0 < t < T$,

$$\sup_{|y|\le\delta} E\bigg[\int_{B_R} \big|u(t,x+y) - u(t,x)\big|\,dx\bigg] \le C_T^R\,\delta^r.$$

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