Consistent Finite Elements for Optimal Control Problems in Computational Fluid Dynamics

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Motivation and governing equations

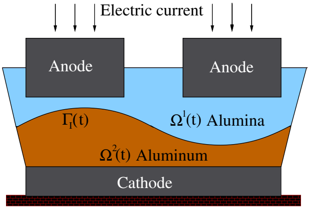

Figure: Aluminum reduction cell

- $\Omega^1(t)$ = liquid Al_2O_3
- $\Omega^2(t)$ = liquid Al
- Temperature: $\sim 950°C$
- Fluids are immiscible
- Formation of an interface $\Gamma_I(t)$

Goal: Track and control the interface position

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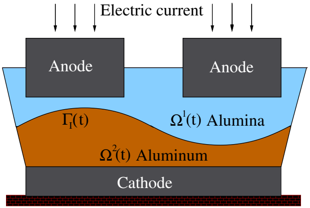

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Goal: Track and control the interface position

Find state $y = (v, p, \rho)$ and control u

$$\min J(y,u), \text{s.t.} \quad \begin{cases} \rho \mathbf{v}_t + \rho[\mathbf{v}\cdot\nabla]\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p = \rho\mathbf{g} + \rho\mathbf{u}, \\ \rho_t + \mathbf{v}\cdot\nabla\rho = 0, \\ \text{div } \mathbf{v} = 0 \\ +B.C.(\mathbf{u}) + I.C. + S.T. \end{cases}$$

Overview

Theory

Optimal control of Oseen equations: (Kiel)

A priori error estimates for SUPG/PSPG stabilized finite elements

Simulation

Phase–field model (Kiel)

Level–set method (Kiel)

Phase–field model in combination with geometric functional

(Tübingen)

Optimal control of Oseen equations

Problems:

- Equal-order FE + small viscosity $\Rightarrow$ stabilization terms
- "optimize-discretize" $\neq$ "discretize-optimize"

Question: What are the differences in terms of accuracy?

Results for SUPG/PSPG stabilized finite elements:

- Optimal order for "optimize-discretize" approach:

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \lesssim \|\mathbf{u} - I_h\mathbf{u}\|_0 + \varepsilon_r(\mathbf{y}(\mathbf{u}_h)) + \varepsilon_l(\mathbf{z}(\mathbf{y}(\mathbf{u}_h)))$$

- Only suboptimal order for "discretize-optimize" approach:

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \lesssim \varepsilon_r(\mathbf{z})+\varepsilon_r(\mathbf{y})+\|\mathbf{u}-I_h\mathbf{u}\|_0+\left(\sum_{K\in\mathcal{T}_h} h_K \|(\mathbf{b}\cdot\nabla)\mathbf{z}^v + \nabla z^p\|_{0;K}^2\right)^{1/2}$$

M. Braack, B.Tews, Linear-quadratic optimal control for the Oseen equations with stabilized finite elements *Tech.rep. University of Kiel, 2011.*

Numerical computations of Navier-Stokes equations

Discretization

- **Time:** implicit Euler scheme
- **Space:** continuous equal order finite elements
- **Stabilization:** LPS for pressure and velocities

Problem: Strong oscillatory behavior when solving

$$\rho_t + \mathbf{v} \cdot \nabla \rho = 0$$

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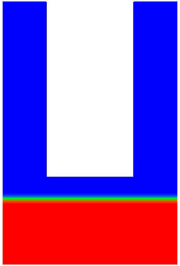

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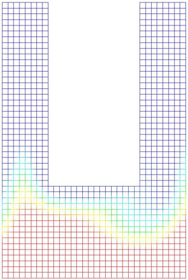

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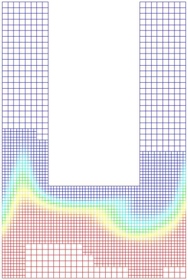

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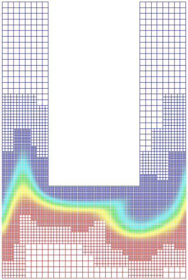

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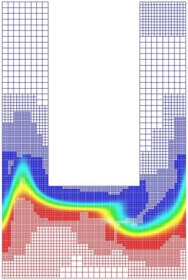

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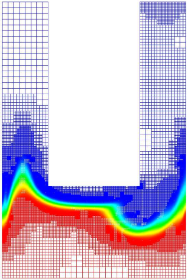
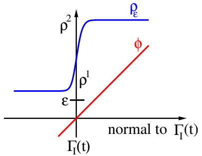
Too diffusive interface even for small mesh size ~ 0.002!



Level-set method as interface model

Osher & Sethian (1988): Interface is described by the zero-level of a higher dimensional and smooth *level-set* Funktion ϕ:



$$\phi(x,t)\begin{cases} <0 & \text{if } x\in\Omega^1(t)\\ =0 & \text{if } x\in\Gamma_I(t)\\ >0 & \text{if } x\in\Omega^2(t)\end{cases} \qquad H_\varepsilon(\phi)=\begin{cases} 1 & \text{if } \phi>\varepsilon\\ \text{smooth} & \text{if } |\phi|\le\varepsilon\\ 0 & \text{if } \phi<-\varepsilon\end{cases}$$

Regularized density: $\rho_\varepsilon(\phi)=\rho_1+(\rho_2-\rho_1)H_\varepsilon(\phi)$

State equation in level-set formulation

$$\begin{aligned} \mathbf{v}_t+[\mathbf{v}\cdot\nabla]\mathbf{v}-\rho_\varepsilon(\phi)^{-1}[\mu\Delta\mathbf{v}-\nabla p+\gamma\mathbb{S}(\phi)] &= \mathbf{g}\\ \phi_t+\mathbf{v}\cdot\nabla\phi &= 0\\ \operatorname{div}\mathbf{v} &= 0\\ \text{signed-distance function to the interface}\quad \phi(0) &= \phi_0 \end{aligned}$$

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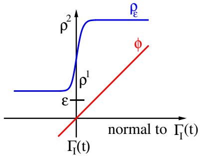

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State equation in level-set formulation $\sigma \sim h_K$

$$\mathbf{v}_t+[\mathbf{v}\cdot\nabla]\mathbf{v}-\rho_\varepsilon(\phi)^{-1}[\mu\Delta\mathbf{v}-\nabla p+\gamma\mathbb{S}(\phi)]=\mathbf{g}$$

$$-\sigma\Delta\phi+\phi_t+\mathbf{v}\cdot\nabla\phi=0$$

$$\operatorname{div}\mathbf{v}=0$$

signed-distance function to the interface $\quad \phi(0)=\phi_0$

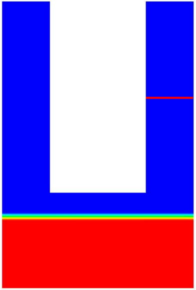

Figure: Domain Ω

Configuration:

- Maximize flow rate of fluid 1 through Γ_{ob}
- Prevent fluid 2 from passing Γ_{ob}
- Observation line:

$$\Gamma_{ob} := \{(x,y) \in \mathbb{R}^2 : y = 1 \quad \text{and} \quad 0.75 \leq x \leq 1\}$$

- Boundary control at inflow part:

$$u = u_0 \sin(\pi t/2) x(x - 1/4), u_0 \in \mathbb{R}$$

- **FEM-Libary: Gascoigne**
- **Optimization toolkit: RoDoBo (Becker, Meidner, Vexler)**

Numerical example

Goal functional:

$$\min J(\phi, u) := \int_0^2 \int_{\Gamma_{ob}} \left\{ (\mathbf{v} \cdot \mathbf{n}_\Gamma)\phi - \log(-H_\varepsilon(\phi) + 1 + 10^{-16}) \right\} ds\, dt + \frac{\alpha}{2} u_0^2,$$

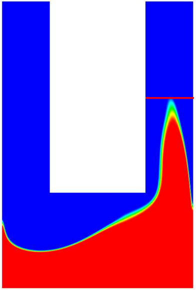

Figure: Density distribution

DWR-Functional: $I(\rho) = \frac{1}{|\Omega_T|} \int_{\Omega_T} \rho_\varepsilon \, d\Omega_T$

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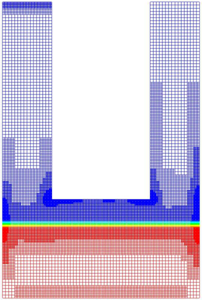

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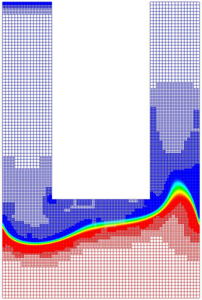

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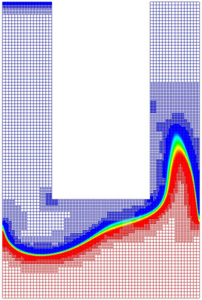

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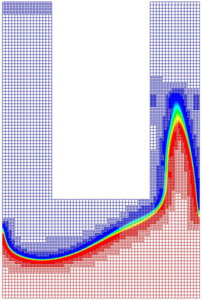

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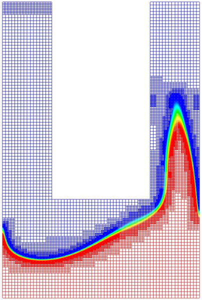

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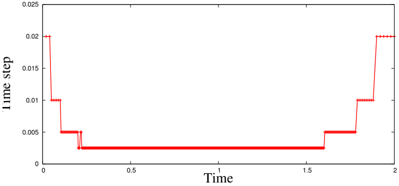

Figure: Adaptive time steps

The Model

Minimize "Shape" "Geometry" "Cost"

$$J_{\delta}(\rho, \boldsymbol{u}) = \|\rho - \sigma\|^2_{L^2(\Omega_T)} + \frac{\beta}{2}\left(\delta\|\nabla\rho\|^2_{L^2(\Omega_T)} + \frac{1}{\delta}\int_{\Omega} W(\rho)\right) + \frac{\alpha}{2}\|\boldsymbol{u}\|^2_{L^2(\Omega_T)}$$

subject to $(\delta, \varepsilon > 0)$

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- Existence and optimality conditions (for $\delta, \varepsilon > 0$)?

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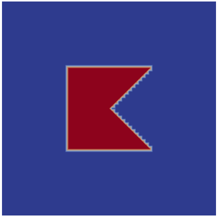
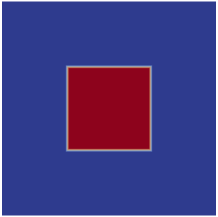
- Behavior for $\delta, \varepsilon \to 0$? **Guess:** $\delta \approx \varepsilon$
- Numerical analysis (stability and convergence) and simulations?

Evidence of the geometric functional

$\|\rho - \sigma\|^2_{L^2(\Omega_T)}$



Target σ



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$+$

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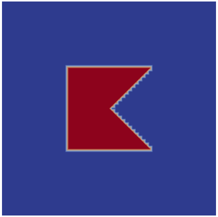

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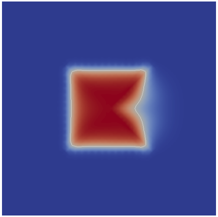

better corners

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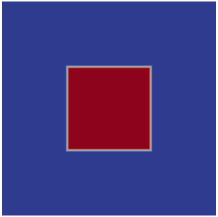

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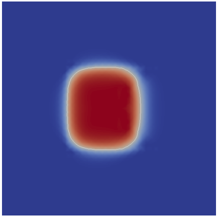
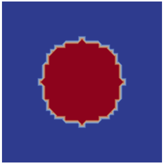


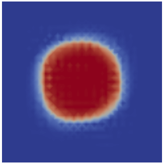
correct geometry

Case $\varepsilon \ll \delta$: parasitic currents

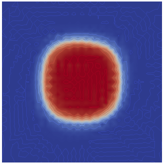
$$\min \; \delta \|\nabla\rho\|^2_{L^2(\Omega_T)} + \frac{1}{\delta}\int_\Omega W(\rho) \quad \text{s.t.} \quad (NSE_\varepsilon).$$



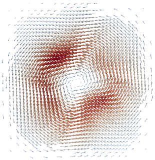
$\rho(t=0)$



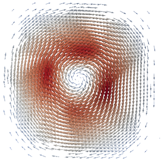
$\rho(t=0.25)$



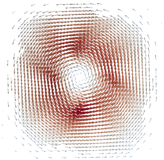
$\rho(t=0.5)$



$\boldsymbol{v}(t=0.05)$



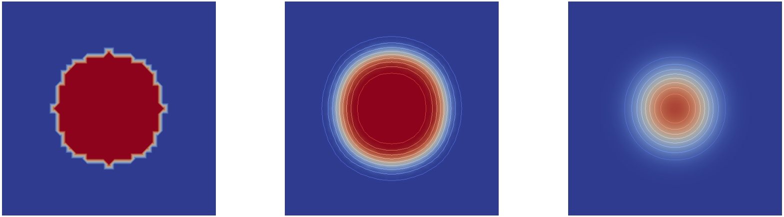
$\boldsymbol{v}(t=0.15)$



$\boldsymbol{v}(t=0.35)$

Case $\varepsilon \gg \delta$: massive diffusion

$$\min \; \delta \|\nabla \rho\|^2_{L^2(\Omega_T)} + \frac{1}{\delta} \int_\Omega W(\rho) \quad \text{s.t.} \quad (NSE_\varepsilon).$$



$\rho(t = 0)$ | $\rho(t = 0.5)$ moderate ε | $\rho(t = 0.5)$ big ε

Strategy and main theorem

Strategy for the discretization

- Fix $\delta, \varepsilon > 0$.
- Use "first discretize, then optimize" ansatz with convergent and unconditionally stable scheme (**Freising 2010**).
- Show existence of discrete optimum, derive discrete optimality conditions.

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Theorem (Convergence)

There exist $\boldsymbol{v}, p, \rho; \boldsymbol{z}, q, \eta; \boldsymbol{u} : \Omega_T \to \mathbb{R}^{(2)}$, such that the solutions of the fully discrete optimality system converge to them in some norms (up to subsequences). The limit functions solve the continuous optimality system.
Moreover, $u_h \to u$ strongly in $L^2(\Omega_T)$ (up to subsequences).

Summary

Kiel

Done

- A priori error analysis (optimal control of Oseen)
- Implementation of a level-set method with re-initialization structures
- **But:** Perturbation of discrete decent direction of optimization solver
- Comparison of phase-field and level-set with adaptivity

Tübingen

Done

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Outlook

- Comparison of the two approaches (Kiel and Tübingen)?
- Optimization with simplified equtions (e.g., thin film eq.)?

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Thank you for your attention!

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